Mathematics

OCT 5 - 1931

THE MATHEMATICS TEACHER

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· OCTOBER · 1931 ·

Volume XXIV · Number 6

THE MATHEMATICS TEACHER

Devoted to the interests of mathematics in Elementary and Secondary Schools

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Correspondence relating to editorial matters, subscriptions, advertisements, and other business matters should be addressed to the editorial office:

THE MATHEMATICS TEACHER

525 WEST 120TH ST., NEW YORK CITY (Editorial Office)

SUBSCRIPTION PRICE $2.00 PER YEAR *(eight numbers)*

Foreign postage, 50 cents per year; Canadian postage, 25 cents per year.

Single copies, 40 cents

PRICE LIST OF REPRINTS

	4pp. 1 to 4	8pp. 5 to 8	12pp. 9 to 12	16pp. 13 to 16	20pp. 17 to 20	24pp. 21 to 24	28pp. 25 to 28	32pp. 29 to 32
50 Copies...	$2.50	$4.00	$6.25	$6.50	$ 8.50	$ 9.75	$11.25	$12.00
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Additional Copies per C.	1.00	1.50	2.50	3.00	3.50	4.50	5.50	6.00

For 500 copies deduct 5%; for 1,000 copies or more deduct 10%.

Covers: For first 50 covers, $2.75; additional 2c each.

Plates printed on one side: For first 50 plates, $2.00; additional 1½c each. Printed on both sides, for first 50 plates $3.00; additional 2½c each.

For more than 32 pages add cost per schedule to make total. Example: for 44 pages add cost for 32 pages and 12 pages.

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THE MATHEMATICS TEACHER

The Official Journal of
The National Council of Teachers of Mathematics
Incorporated 1928

OCTOBER 1931

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Volume XXIV **Number 6**

Published by the
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
Menasha, Wisconsin New York

Entered as second-class matter at the post office at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in paragraph 4, section 412 P. L. & R. Authorized March 1, 1930.

THE MATHEMATICS TEACHER is published monthly except June, July, August and September. The subscription price is $2.00 per year. Single copies sell at 40 cents each.

F. Dupin Sculp.
MARIN MERSENNE
Religieux de l'Ordre des Minimes Thologi
Philosophe et Mathematicien celebre né a
Oyse au Maine Mort a Paris 1648. âgé de 60 ans.

THE MATHEMATICS TEACHER

Volume XXIV Number 6

Edited by William David Reeve

Some Simple Logical Notions Encountered in Elementary Mathematics

I

NECESSARY, SUFFICIENT, AND NECESSARY AND SUFFICIENT CONDITIONS[1]

By CARL A. GARABEDIAN

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THE OBJECT of our first paper is to explain the logical notions suggested by the title. The caption is imposing; but the notions are met very early in the study of mathematics—as well as in the incidental mental transactions of everyday life—and yet they are rarely explained in mathematical literature.[2] Perhaps this discussion may arouse some interest in adequately presenting these important concepts to the student.

[1] Part II, "On proving identities and solving equations," will appear in the November issue of this journal. Both papers are developed from a talk given to the Harvard Mathematics Club on December 15, 1920.

[2] The only text in which I have found an explanation of the terms in question is Osgood's *Introduction to the Calculus,* Macmillan, 1922, p. 71. The terms are also briefly explained in Couturat's *Algebra of Logic,* Open Court, 1914, p. 7, and in Keyser's *The Pastures of Wonder,* Columbia University Press, 1929, p. 112.

We confine ourselves in this presentation to examples chosen from elementary mathematics. Consider, from plane geometry, the following theorem: "If a triangle is equilateral, then the triangle is isosceles." This straightforward theorem is typical of many of the theorems of mathematics in that the statement is essentially of the form:

(1) "If *E*, then *I*."

Let us examine the structure of form (1). To simplify the discussion, we denote the "condition" that a triangle be equilateral by the single letter "*E*," and the condition that a triangle be isosceles by the single letter "*I*." Although we purposely choose letters which help us interpret form (1) in terms of our illustrative example, it should be noted that (1) is a general form of statement and that "*E*" and "*I*" need not refer specifically to the conditions with which our illustrative theorem is concerned. Form (1) represents a certain relation existing between two conditions. This relation is clearly one of implication; its meaning is satisfactorily explained by using the words "if" and "then." Similarly, we explain what is meant by a "condition" by merely giving examples. We may call a theorem of the form (1), in which one condition implies another condition, a "proposition."

In the light of (1), our theorem from plane geometry appears as a particular example of a very general form of proposition. Had our theorem read: "An equilateral triangle is isosceles," or merely: "Equilateral triangles are isosceles," it would still be quite obvious that the theorem is essentially of the form (1); yet it can happen that the statement of a theorem or proposition is reducible to form (1) only after the exercise of some ingenuity.

We can now define the terms "sufficient condition" and "necessary condition." A *useful* definition is the following. The statements

(1) "If *E*, then *I*,"
(2) "*E* is a sufficient condition for *I*,"
(3) "*I* is a necessary condition for *E*,"

are coextensive; that is, any one of these statements implies the other two—in short, all three statements are equivalent. Hence, to say that "A sufficient condition that a triangle be isosceles is that the triangle be equilateral," or that "A necessary condition that a triangle be equilateral is that the triangle be isosceles," is the same as

saying that "If a triangle is equilateral, then the triangle is isosceles."[3]

But we have not yet gathered together all the material relevant to the present discussion. Consider next the following theorems: "If a triangle is equilateral, then the triangle is isosceles," and "If a triangle is not isosceles, then the triangle is not equilateral." From the fact that these two theorems are coextensive, it appears that our illustrative theorem has still another equivalent form; and it is precisely this new form which furnishes an explanation of what shall be meant by the "negative" of a condition. We shall denote the negative of the conditions "*E*" and "*I*" by "*Not-E*" and "*Not-I*," respectively. Furthermore, by the negative of "*Not-E*" shall be meant "*E*" itself.

The equivalence of the two theorems just given suggests the equivalence of the propositional forms:

(1) "If *E*, then *I*,"

(4) "If *not-I*, then *not-E*."

Observe that it is not true either that "If a triangle is isosceles, it is equilateral," or that "If a triangle is not equilateral, it is not isosceles." Thus our illustrative theorem shows us that we cannot infer from (1) and (4) either that

(5) "If *I*, then *E*,"

or that

(6) "If *not-E*, then *not-I*."

The inexperienced may find this last fact disconcerting. For example, a friend says: "If it doesn't rain, I'll go to the game with you." But it cannot be inferred that if it does rain, the friend won't go to the game; the first statement does not undertake to decide the friend's course of action in the event of wet weather. To be sure, if it does rain, the friend may decide not to go to the game; on the other hand, he may go to the game in spite of the rain. Of course it can happen that we prefer to employ, for emphasis, negatives which do not alter the logical content of a statement; for example, it is a question irrelevant to the present paper whether we choose to say: "I should like to do something about it," or "I should not wish to do nothing about it."

Let us now forget for the moment our illustrative example, and,

[3] Note that our new versions of form (1)—that is, (2) and (3)—result from a renaming of the relationship of implication together with an explicit characterization for the *inverse* of the relationship.

recalling only our first definition, ask what it shall mean to say that

(7) "*A* is necessary and sufficient for *B*,"

where the letters "*A*" and "*B*" again denote conditions. Here the natural interpretation is that not only

(8) "*A* is sufficient for *B*,"

but also

(9) "*A* is necessary for *B*."

Indeed, our definition of "necessary and sufficient" shall be that (8) and (9) hold simultaneously; that is, we take the combination (8) and (9) to be coextensive with (7).

From our first definition, we see that (8) and (9) are coextensive with

(10) "If *A*, then *B*,"

and

(11) "If *B*, then *A*."

But in (10) and (11) we recognize a theorem and its converse; that is, if either statement be taken as the theorem, then the other statement is what is technically termed the *converse* of the theorem.[4] And herein lies the reason, so often vague to the beginner, why, in order to prove that a condition is both necessary and sufficient—(8) and (9)—a mathematician proves a theorem and its converse—(10) and (11). Or he may find it desirable to prove both that "*A* is necessary for *B*," and that "*B* is necessary for *A*." Or, again, he may find it convenient to prove that both "If *A*, then *B*," and "If *not-A*, then *not-B*" hold simultaneously. Let it also be made clear to the student that when a writer begins a theorem with "If and only if," or with "When and only when," he is signalling to the reader the fact that he is about to enunciate really two theorems in one, namely, a theorem and its converse.

We see further, again by means of our first definition, that "*A* is necessary and sufficient for *B*" is coextensive with "*B* is necessary and sufficient for *A*." This fact prompts us to call conditions *A* and *B* "equivalent" when either is necessary and sufficient for the other. Observe that from "*A* is necessary and sufficient for *B*" and "*C* is necessary and sufficient for *B*" it may be inferred that *A* and *C* are equivalent conditions.

[4] Note that our illustrative theorem is one whose converse is not true.

All these matters seem clear, perhaps, but the facts are not so easy to apply in practice—since the mathematics is only too often so involved that the simple issue is obscured. The reader's difficulty lies, therefore, not so much in being able to comprehend as in being able to recognize. In this connection, it is not pedantic to remark that mathematicians often talk about *the* necessary and sufficient condition when, of course, accuracy of statement demands the indefinite article. We may have a long string of equivalent conditions that yet have wholly different meanings. For example, what is the condition necessary and sufficient that a triangle be isosceles? Is it that the triangle have two equal angles, or that it have two equal sides, or that it have two equal angle bisectors—or what? And the fact that, in some given instance, the string of equivalent conditions does not suggest itself—or is of no importance—does not mean that such a string does not exist. A few years ago, at a large gathering of scientists in Washington, a prominent geologist advanced a new theory of isostasy only to have it rather generally ridiculed because of the wide acceptance of Hayford's theory. Many geologists were under the impression that *the* theory had been found! Are not misunderstandings of this sort encountered constantly in the applied sciences?

While on this subject of pitfalls and snares, we should like to mention two further instances which illustrate our point. In Murray's *Differential Equations*, Longmans, Green, 1914, we find on page 134 the paragraph: "Therefore, in order that $u = a$ may be an integral of (1), it is necessary and sufficient that $u = a$ be an integral of the system (2) Art. 99, *and conversely*." The italics are ours. And in Carslaw's *Fourier's Series and Integrals and the Mathematical Theory of the Conduction of Heat*, Macmillan, 1906, we read[5] at the bottom of page 58 the theorem: "Uniform convergence is thus a *necessary* condition for continuity. It is not a *sufficient* one, since different non-uniformly convergent series are known which represent continuous functions in the interval of non-uniform convergence." It is possible, certainly in the first of these examples and conceivably in the second, to detect the error even without knowledge of the subject matter involved.

It may be remarked that a theorem and its converse, (1) and (5),

[5] This example was called to my attention by Professor W. C. Graustein. The mistake is corrected in the second and third editions, 1921 and 1930.

have also certain equivalent negative forms, (4) and (6); and application of this fact may be made not only to (10) and (11) but also to (8) and (9). But we forego further elaboration and pass on to another aspect of our subject which deserves some mention before a discussion of this elementary nature is brought to a close.

Throughout mathematics, the conditions we have been studying appear in the form of tests. These it is natural to classify as "necessity," "sufficiency," and "necessity and sufficiency" tests.

To take a simple example, a sufficiency test to determine if a triangle be isosceles is to test it for the equilateral property.[6] We now observe that if this type of test works—that is, if the sufficient condition is fulfilled—then we know that the triangle is isosceles. But suppose that the sufficiency test is not fulfilled; then it is important to note that the knowledge that the sufficiency test is not fulfilled carries with it no information that bears on the question of whether or not the triangle is isosceles. It is true that we know that our triangle is not equilateral, but this incidental information is, for the question at issue, irrelevant. In other words, we have not been able to show that our triangle is isosceles; neither have we any right to infer that the triangle is not isosceles. Obviously, a triangle may be not equilateral and yet be isosceles. The fact we have been illustrating is of such importance that we embody the general result in a theorem:

Theorem I. A sufficiency test tells something when the sufficient condition is fulfilled; but the test fails when the sufficient condition is not fulfilled.

Let us pass on to the necessity test. A necessity test to determine whether or not a triangle is equilateral is to test it for the isosceles property. How does this test help us? If the triangle *is* isosceles, we have, it is true, no information on the point at stake. We cannot infer that the triangle is equilateral; neither are we able to show that the triangle is not equilateral. But suppose that the test condition is not

[6] The situation may be visualized by means of a figure consisting of two circles, one lying wholly within the other. Let the points lying within the smaller circle denote triangles which are *E* and the points lying within the larger circle denote triangles which are *I*. Then points lying within both circles represent triangles which are both *E* and *I;* points within the larger circle but not within the smaller circle represent triangles which are *I* but not *E*. Many teachers will wish to use this type of diagram for classroom presentation.

fulfilled, namely, that our triangle is found to be not isosceles. Are we any better off for this information? Yes, for we now see that our triangle cannot possibly be equilateral. We argue that if it were equilateral, it would also be isosceles; and this contradicts the fact, just established, that the triangle is not isosceles. We have a second important result, which we state, more generally, as a theorem:

Theorem II. A necessity test tells something when the necessary condition is not fulfilled; but the test fails when the necessary condition is fulfilled.

Simple as the situation is here, it may easily be obscured. We find an example in point on page 28 of Bachmann's *Grundlehren der Neueren Zahlentheorie,* Second Edition, de Gruyter, 1921. Here is a theorem in which, apparently, a conclusion is drawn from the fulfillment of a necessary condition; but examination of antecedent text material soon makes it clear that the condition, in this instance, is also sufficient—and that it is from the fulfillment of the sufficient condition that the conclusion of the theorem is inferred. The confusion seems to be due to an unfortunate use of the term "necessary condition" in a non-technical sense, namely, in the sense of a sufficient condition which must *necessarily* be fulfilled if a certain conclusion is to follow. It is precisely because ambiguities of this sort occur in the literature that the reader does well to be informed on correct terminology.

Finally, from obvious considerations we have also the theorem:

Theorem III. A necessity and sufficiency test is a complete test in the sense that it yields one conclusion when the condition is fulfilled and the opposite conclusion when the condition is not fulfilled.

A necessity and sufficiency test for an isosceles triangle is the test for two equal angles. A necessity and sufficiency test for an equilateral triangle is the test for three equal angles.

In practice, all three types of test we have described are met frequently. It is important, therefore, that the student work out for himself the principles involved. Problem material from elementary mathematics is plentiful; but, for the student who is sufficiently advanced, perhaps the richest field for experimentation in these matters is the topic of relative maxima and minima, at the beginning of the calculus. Moreover, additional information comes to light when theory is put into practice; for example, in the applications, it can

happen that a complete test—because it "tells too much," so to speak —is less useful than a test which is merely sufficient or merely necessary.

It is clear that the student who has only a dictionary to enlighten him on the meanings of the words "necessary" and "sufficient" will hardly arrive readily at a proper understanding of the technical subject we have been elaborating. Let the teacher, then, help explain these notions, which—being incidental tools borrowed from logic—have become a part of the mathematical vocabulary without appropriate elucidation.

News Notes

During the winter of 1930-1931, Section 19 of the New York Society for the Experimental Study of Education has had eight dinner meetings at the Men's Faculty Club of Columbia University with an average attendance of 106 at the dinners and of 183 for the discussions which follow. The officers for 1931-1932 are:

Chairman, Professor W. D. Reeve, Teachers College, Columbia University.

Assistant chairman, Professor W. S. Schlauch, New York University.

Secretary, Alma Ekholm, Girls Commercial High School, Brooklyn.

The following outline shows the program of the past year:

October 25, Professor D. E. Smith, "The Play of the Imagination in Geometry."

November 22, Professor W. S. Schlauch, "World Trends and Forces, A.D. 1931."

December 20, Dr. Vera Sanford, "The Use of the History of Mathematics in the High School."

January 24, Einstein Film on Relativity.

February 28, Professor W. S. Schlauch, 'Report of the Meeting of the National Council of Teachers of Mathematics held at Detroit, February, 1931."

March 28, Professor Cassius J. Keyser, "Educational Values of Plane and Solid Geometry."

April 25, Dr. Dehn, "Graphical Charts for Solution of High School and College Mathematics"; Professor W. S. Schlauch, "Demonstration of a Slide Rule for the Solution of Quadratic Equations."

May 23, Dr. Tildsley, District Superintendent of New York High Schools, and Professor David Eugene Smith, "Mathematics in the High School."

Mathematics in the Home

By KATHARINE BALL PALMER
Tenafly, New Jersey

AN ANALYSIS OF the use of mathematics in the modern home, particularly in urban communities, reveals the extent of the changes that have occurred in women's home activities as a result of the increase in variety and excellence of factory production. The work that was typical of the home of former days,—garment making, knitting, canning, baking, laundering, and fine sewing—has gradually been taken over by shop, factory, commercial and tax-supported organizations. Housework has been lightened by conveniences unknown to former generations, by homes designed to save labor, by machines for cleaning, sewing, and washing, by electrical equipment of every variety, and by fuel that needs only the effort of lighting a match or turning a switch.

Preparation for home-making must obviously be adapted to meet these changed conditions. Skill in cooking, sewing, and cleaning evidently falls short of adequacy in homes where a large part of this work is done outside or by machines. Women who now buy the things that used to be made in the home need to acquire skill in planning how much they can afford to pay for their home necessities and in learning how to buy economically quite as much as they need skill in the traditional home occupations of cooking and sewing.

Knowledge of the kind of food to buy has become quite as important as the ability to prepare it. Discrimination in purchasing garments as to quality of materials and cost of manufacture is even more important than skill in making them. The emphasis has changed. Women have become, not producers but consumers. They are today the most important group of retail buyers.

The economic status of women is changing also. They have become wage-earners. Even their work in the home is recognized as a definite addition to the family income. Their time has a money value. They have a share in the responsibility for planning the family budget, and as the purchaser for the family, they are largely responsible for the success of the family in living within its limitations.

The extent to which mathematics is used in the modern home is

not so significant a subject for consideration as the extent to which mathematics might be serviceable in equipping women for more effective home-making. Comparatively little investigation of this question has been made, and almost no educational experiments have been carried on. Text books in arithmetic for the grades have included an increasing number of problems based on home topics, but these seem rather to illustrate some arithmetical principle than to promote intelligent reasoning about home activities. If specific skills are needed, or if mathematics can contribute to more effective home management, the fact would have a definite bearing upon curriculum planning.

That serious difficulties are met in home-making is only too apparent from the wasteful and uneconomical methods of the average home. An attempt is being made through the federal aid for schools that conform to the Smith-Hughes law, to remedy the situation by the required courses in home economics in the public schools. These courses include much more than cooking and sewing although they often appear under these names in the school curricula. It is possible, if the trend towards centralized production increases, that even greater emphasis will be placed upon the economic aspects of preparation for home making, and that more attention will be given to budget making, keeping accounts, methods of purchasing, and to the analysis of underlying sociological and economic conditions. For these phases of home economics, something in mathematics ought certainly to contribute valuable preparation in the acquisition of skill in quantitative thinking and in the development of ability to make logical analyses and correct deductions.

Classification of Home Problems

The phases of family life are often grouped under the conventional headings of the typical family budget—food, shelter, clothing, operation, and the higher life. But, since the point of view in this discussion is rather the application of mathematics to home situations than the study of home management, it might be more natural to consider together those problems that depend upon the same mathematical principle. This is the usual method of the arithmetic text book, e.g., decimals and their application, percentage and problems in percentage. This method of grouping tends to obscure the continuity of purpose that is inherent in home-making.

To emphasize this unity of purpose, and at the same time to in-

dicate how necessary it is for the home-maker to have a knowledge of mathematics that shall "function" in practical situations, the problems selected for this brief presentation of the use of mathematics in the home have been grouped around the three principal duties of the home-maker:

1. The Budget
2. Providing for the equipment and maintenance of the home.
3. Utilization of public service.

The Budget. Economic independence is comparatively new in the experience of women. Their new economic status tends to increase their responsibility in planning the family budget. The modern home, especially the urban home of limited resources, requires skill in planning to meet the demands of city life. Budget-making and the keeping of accounts are as necessary for a successfully managed home as for business. The family has definite minimum needs for food, clothing, shelter, operation and for the higher life. Abundant helpful information is available for the woman who understands how to apply it to her own immediate problems.

The most important task in connection with budget-making is to decide upon a system of keeping the family or the personal account. Some kind of an account is imperative. If the expenditures are classified according to the budget, the accounts are more useful, not only in determining daily questions as they arise, but also in furnishing data in a convenient form for modifying budgets from year to year, or even from month to month. Here are some illustrative samples:

1. A man who works on a commission finds that his receipts for five years were as follows: $2,100, $1,800, $1,650, $2,000, and $2,200. What amount of income would be advisable to use as the basis of the family budget? Using the family accounts determine the per cent of the income that has been used for each of the five divisions of the budget. With these figures as a basis, make a tentative budget for the coming year.

2. A teacher is paid in ten installments of $225 each. How much should be set aside each month to provide for the summer vacation? Can she afford an automobile?

3. A family own their own home which is valued at $4,000. They raise vegetables to the amount of $250, and in addition to this they have an income of $1,200. They pay $80 for insurance and taxes. If property in this locality rents for 10 per cent of its value, allowing for repairs, taxes, etc., how much gross income does the house theoretically add to the family income? How much net income? What should be the family budget?

4. Two positions are offered—one that pays $45.00 a week but is subject to seasonal "lay-offs" of from six to ten weeks every spring, and another that pays $2,000 a year. If the first position is accepted, how should the budget be adapted to meet the weeks of loss of income? What consideration might make the second position less desirable? If the lay-off should be ten weeks, what would be the average weekly wage in each?

5. A salesman whose salary is $125 per month wishes to rent a house in his community How much can he afford to pay?

6. How much can a salesgirl whose weekly wages are $22 afford to pay a week for board and lodging? For clothes per year?

7. A woman can buy a house and lot suitable for her family for $3,200 or she can rent a house for $20 a month and invest her money in bonds paying 5½ per cent interest. It will cost $60 a year to keep the house in repair. The tax rate based on 75 per cent of the value of the property is 7½ mills on a dollar. She would insure the house for $2,500 at 45 cents per $100 for one year. What would be the most economical thing for her to do?

8. A family has been paying $24 a month for rent. How expensive a house can they afford to purchase?

9. A family has an income of $2,800 from the father's salary and in addition the interest on a $1,000 bond at 4½ per cent. How much can they afford to invest in a home?

10. A woman who has been paying $4.00 a week for washing finds that she can buy a washing machine for $115, that will make it possible for her to do her own work. The machine uses 250 Watts per hour, the local rate for electricity is 7 cents per KWH, and she will use the machine four hours each week. How long will it take her to save the cost of the machine? How much will the current cost? If she rents the machine to her neighbor in the next apartment, what would be a fair rate to charge?

11. What is the cost of a 55-Watt electric iron that is used 8 hours every week if the local rate is 8 cents per KWH?

12. Compare the cost of service by the hour at 30 cents including lunch valued at 15 cents, with the cost of service by the week at $8, including board at $5.50. Make the comparison on the basis of an 8-hour day.

13. A woman spends 42 hours a week preparing food for the family and doing the rest of the house work. What does she add to the annual family income by this service?

14. How much can a family of four whose income is $2,800 afford to pay a week for a maid and for care of the furnace, lawn, etc.? Can they afford a car? A radio? A washing machine?

15. A woman's home is valued at $3,750. Her income is $1,200. How much may she allow in her budget for service?

16. Can a family with an income of $1,500 afford to pay $2 a week for laundry and cleaning, if this is the only expense for service? How much should be set aside for savings? for recreation and the higher life?

17. A woman works 34 hours a week preparing meals and clearing them away, 9 hours buying, making and repairing clothes, 12 hours on laundry

work, 9 hours on cleaning, 10 hours looking after the children, and 3 hours in planning and management. Mary, age 12, works 3 hours a week, and John, age 10, works 2 hours a week. Consider the value of Mary's work at 10 cents an hour, and John's at 8 cents an hour. How much is added to the family income by the work of these three?

18. What percentage of the time does the mother spend on each kind of work? How much does her work increase the value of the food materials purchased? Of the materials for clothing and the clothing purchased?

19. A home had a large old-fashioned kitchen in which much time was wasted because of the distances between the sink, cupboards, and stove. After the kitchen had been remodeled at a cost of $85, the time required for preparing a meal was cut about ten minutes. How much time was saved in a year? If the cook received 40 cents an hour, what was the amount of saving? How long would it take to pay for the cost of alterations?

20. What would a woman 35 years old have to save annually if she invested it at 4 per cent compound interest, in order to have her savings amount to $10,000 by the time she is 60 years old?

21. A man has partially provided for his family in case of his death by taking out a life insurance policy for $5,000 on which he pays an annual premium of $125. If he can invest his additional savings so as to bring 4½ per cent compound interest, how much more would he have to save annually to provide another $5,000 by the end of 20 years?

22. Would you advise a husband to put all his savings into life insurance?

23. If you have no one dependent on you in any way for support and no unsecured debts, would you invest in life insurance?

24. Under what conditions would you advise an unmarried woman to invest in life insurance?

25. A teacher wishes to retire when she becomes 60 years old. How much ought she to save in order to provide for the cost of a $1,200 annuity at that time, if the rate for a woman 60 years old is $1,174.60 per $100?

Supplying the Needs of the Family. Women have become the most important group of retail buyers. The food and garments formerly made at home are now offered for sale at prices with which the home cannot compete. To buy skillfully requires more than the appreciation of workmanship, design, and quality of materials. A good buyer must know how to take advantage of a favorable market trend, to understand the cost of credit, and of service, to distinguish between actual and relative differences in prices, and to know the relation between the cost of the raw materials and the finished product. The buyer must be able to estimate her needs accurately, and to determine whether or not her needs can be supplied more economically by home manufacture or by purchase.

Scientific knowledge of foods is essential. The selection of the right

kind of food to secure adequate amounts of the various nutritive elements is of at least as much importance as skill in cooking. Both wage-earning women and home-makers must understand how much and what kind of food is required for healthful living. Well-cooked foods can be bought, but well-planned dietaries require scientific knowledge. When the family allowance for food is limited, the task of planning the dietary challenges even an expert dietitian. Bringing children up to normal weight, preventing unwholesome increases in weight, altering a diet by making correct substitutions of cheaper or more appetizing foods that contain the same essential nutritive elements, altering the diet for special physiological conditions, and similar dietary problems involve not only scientific knowledge of food but the ability to make quantitative judgments and accurate computations. Here are some typical problems:

1. Much of the scientific data in regard to dietaries and nutrition is stated in terms of the metric system. In a scientific report the normal weight of a six-months baby was said to be 6.8 kilograms. Is a baby that weighs 15 pounds up to normal?

2. How many calories per day are required by a general house-worker, who weighs 152 pounds and whose daily schedule is as follows: Sleeping, 8 hours; sitting, 4 hours; at active exercise, 8 hours; at light exercise, 4 hours?

3. A family of 2 adults and 5 children is allowed $12.50 per week for food. If their total energy requirement per day is 13,000 calories, what is the average allowance per 1,000 calories? Per 100 calories?

4. How much may a family of four, whose income is $1,200 a year, spend per year for food? How much per month? Per day? If the daily fuel requirement is 11,000 calories, what should be the average cost per 100 calories?

5. How many pounds of rib roast beef will it take to equal one dozen eggs in fuel value? Compare the cost if roast beef costs 35 cents a pound and eggs cost 40 cents a dozen; 55 cents a dozen; 60 cents a dozen; 80 cents a dozen.

6. Compare proteins, carbohydrates, and fats as sources of fuel.

7. A man was advised by his physician to reduce his daily diet from 3,000 to 2,500 calories. If the diet is not to be reduced in bulk, what kind of substitutions can be made? What alteration is effected by serving tomatoes instead of peas? Oysters instead of pork chops? What reducation in calories is made by leaving out cream and sugar in coffee, if he was accustomed to having two teaspoonfuls of sugar and one teaspoonful of heavy cream in his coffee three times a day? What reduction is made if butter is omitted when he had been accustomed to having one ounce at each meal? If one slice of bread is allowed per meal instead of three?

8. How much sweet chocolate will it take to yield as many calories as a 10-ounce box of soda crackers and 1/8 pound of cheese? What is the difference in cost?

9. Select from the table of 100-calorie portions five foods in which approximately the same number of calories is supplied by protein and compare the cost of these foods per 100-calorie portion.

10. What foods are comparatively economical sources of iron? Of calcium? Of phosphorous? Of vitamins?

11. Compare tomatoes and oranges as to their cost as sources of vitamins.

12. If peanuts are substituted for 2½ pounds of round steak, how many pounds must be bought to produce the same number of calories? What is the saving in using peanuts at the local prices?

13. Plan a day's dietary for a child of nine at a cost not to exceed 40 cents.

14. If not more than 15 per cent of the food budget should be spent for meat, poultry, and fish, examine the food bills of the family for a month to see whether they conform to this standard.

15. If from 25-30 per cent of the food budget is allowed for milk, how many quarts a month may be bought on a food budget of $37 when milk is 17 cents a quart?

16. Alter the following recipe for pie crust to make one half the amount, and state in the most convenient measures:

Flour, 2 cups; lard, 3/8 cup; baking powder 1/2 teaspoonful; salt, 1 teaspoonful; ice water, 1/2 cup.

17. Alter this recipe to serve two persons:

Rice pudding for 6 persons; 1/2 c. of rice, 1/4 ts. of salt, 1/3 c. of sugar, spk. grated nutmeg, 1 qt. hot milk.

18. The price of sugar increased from 6 to 9½ cents a pound. What is the actual increase per 25 pounds? per 100 pounds? What is the percentage of increase?

19. If other household commodities increased in price at the same rate, what would be the increase in the budget for food for a family that expected to buy their food for $35 a month?

20. If walnuts are 30 cents a pound and are 58 per cent refuse (shells), what is the cost of one pound of meats? If shelled walnuts sell for $1.00 a pound, which is the cheaper way to buy walnuts?

21. If a breakfast cereal costs 10 cents a box or $1.00 per case of 12 boxes, what is the percentage of saving in buying it by the case?

22. Compare the cost of canned salmon at the following prices: 23 cents, net weight 14 ounces; or 18 cents, net weight 10 ounces.

23. Determine which brand of peas is the more economical, the weight of each being 20 ounces per can:

Volume of liquid.	Price per can.
7 oz.	20 cents
8 oz.	25 cents
4½ oz.	15 cents
(Vacuum packed No liquid)	

24. Crêpe 40 inches wide can be bought for $2.25 per yard. 36 inch material of approximately the same quality costs $2.00. If the narrower material is wide enough for a garment 56 inches long that takes three length of goods, what is the amount of saving by buying the narrow material? What is the percentage of saving?

25. If a garment can be cut from 3½ yards of 40-inch material but requires 4½ yards of 36-inch goods, which is cheaper, 36-inch silk at $2.50, or 40-inch silk at $2.85.

26. What is the percentage of saving in buying handkerchiefs 2 for 25 cents over buying them at 15 cents apiece?

27. A woman spends 10 hours making a dress for her daughter, the materials for which cost $2.50. Estimate the value of the completed dress if the local rate for sewing is 50 cents an hour.

28. A ready-made dress can be bought for $15.00 or a similar dress could be made at home by a mother in 16 hours. If a woman would have to be hired to do the housework during this time at $.35 an hour, and the materials for the dress would cost $9.75, which is the cheaper plan?

29. If it takes 8 hours to make a middy blouse, estimate the cost of the material and the cost of labor for making, and compare the total estimate with the cost of a ready-made middy blouse of approximately the same quality.

30. A child's dress is 2 yards around the bottom. How much lawn 1 yard wide is required for a full-shirred ruffle 7½ inches deep finished with a half-inch hem and three eighth-inch tucks?

31. Estimate the number of yards of dotted swiss, 32 inches wide, needed for 4 curtains each 4 feet 6 inches long, trimmed along one side and across the bottom with a 3-inch ruffle finished with a ¼-inch hem and set on with a ⅜-inch receiving tuck. Allow 3 inches at the top of each curtain for the rod and heading and 3 inches for shrinking.

32. How many strips of bias need to be cut from a strip of silk 10 inches wide to make 4 yards of bias binding?

33. How many ¼-inch tucks are needed to shorten a garment 7 inches?

34. How many yards of real lace at $2.25 a yard, are required to edge a circular lunch cloth for a table 54 inches in diameter?

35. In buying plaid for a kilted skirt, a girl bought 4 yards when 3½ yards would have been enough. If the extra ½ yard could not be used, what was the percentage of increase in the cost of material due to inaccurate estimating?

36. Estimate the cost of inaccuracy if 5 yards of chiffon at $2.50 per yard are purchased for a dress that requires only 3½ yards.

37. Through carelessness in measuring the windows, a woman lacked 10 inches of having enough material for the fourth window of the dining room, and had to buy 2½ yards more net at 83 cents a yard. Find the cost of her carelessness.

38. The yearly clothing budget allows $350 for a family consisting of the following members: father, mother, a lad, and two daughters. Apportion the

allowance among the different members of the family. Make clothing budgets for each on a three-year basis.

39. In a family there are three children: 2 girls aged 12 and 3, and a boy, aged 10. The family income is $2,300. If 20 per cent of this is allowed for clothing, how much would you allow for each member of the family?

40. A man and his wife have an income of $2,800. They plan to allow 15 per cent of this amount for the clothing for themselves and their children, a boy of 6 and a girl of 14. How shall the money be divided among them?

In the foregoing groups of home problems it is evident that some skill in arithmetical computation, in quantitative thinking, and in logical analysis would be useful in the home and also, it may be added, in the personal life of the self-supporting whose personal problems are practically the same as those of the typical home-maker. Several questions present themselves. Is the right kind of preparation in mathematics being offered in the schools? What is the right kind of training in mathematics? Are algebra and geometry useful? If special courses are desirable, when and by whom should they be given? And to whom?

In view of the wide spread dissatisfaction with the present mathematics courses on the part of the general public, and the increasing tendency to drop algebra and geometry as required courses, even to allow students to enter college without the time-honored sanction of a passing grade in algebra at least, these questions should give mathematics teachers cause for thought. If specific skills are valuable to women, if the analytic method of reasoning developed in algebra and geometry can be developed so as to "function" in daily life, these facts are of importance to those who plan curricula.

Before the questions can be answered, more information is necessary, and more experimental data.

The Family in Relation to the Community. Many responsibilities of the home have been taken over by the community or by commerical organizations. Among these may be mentioned health supervision, garbage and sewage disposal, street cleaning, water supply, as well as the distribution of gas and electric current. Many cities also maintain amusement parks, playgrounds, swimming pools, and skating rinks. The relative cost of these forms of public service is a matter of concern to every family. The modern home-maker who is also a voter needs to be able to understand the way in which this public business is managed and paid for, and whether or not her family is getting its money's worth. Many phases of these matters

come before the public for decision at elections, and women as voters should be able to understand the social and economic problems that lie back of their gas, water, and tax bills. The following typical problems will illustrate the possibilities:

1. A neighborhood club of 12 families is organized for outdoor sports. They decide to purchase, during the first five years, a croquet set, a tennis net, a volleyball, and a set of standards for volleyball, field hockey goals, 2 basketballs, and a set of goals. What is the total cost of the equipment? If an outlay of $3 per year will keep the equipment in repair for 5 years, what is the average annual budget for equipment? How much should each family pay each year?

2. An investigation in 14 cities shows that of 33,122 children, 45 per cent were "loafing" outside of school because, as they said, there was "nothing to do" How many children in these cities were not gaining the benefit of play?

3. In the city of Minneapolis in 1918 the cost of repairs and maintenance for the city parks and recreational facilities, including school playgrounds, public parks, baths, and public playgrounds, was $322,000. If the population was about 400,000 what was the outlay per capita?

4. Compare the park area per person in Newark, New Jersey, whose population was 399,000 and park acreage 14,858; and Portland, Oregon, whose population was 292,000 and park acreage 37,555. (Figures from 1916 census of cities.)

5. If a family income is $1,300, how much better chance has a baby to survive the first year in Montclair, New Jersey, than a baby whose family has an income of $800?

Births and deaths under 1 year, according to total family income
(See U. S. Children's Bureau)

Total family income	Births	Deaths under 1 year
Under $625	95	11
$625 to $1,199	111	9
$1,200 and over	128	6

6. The rate for electric current is decreased from 9 cents to 7½ cents per KWH. What is the per cent of decrease? If the consumption per month had been approximately 960 KWH what is the amount of decrease in the budget?

7. Typhoid fever due to impure water caused 15 of the 95 deaths in 1928 in a town whose population numbered 3680. What was the death rate per 1,000 from all causes? The death rate due to typhoid fever? If the average productive value of a life is $1,700, what was the loss of productive value in this town due to this preventable cause?

8. Through improved methods of control, the death rate from diphtheria in Massachusetts dropped from 13.5 per 1,000 in 1880 to 2.4 in 1908. In a community of 75,000 how great a saving of life was effected?

9. There are approximately 30,000,000 wage-earners in the United States. It has been estimated that they lose an average of 9 days each year on account of sickness. If the average wage is estimated at $2.50 per day, and the average

cost of medical attention at 75 cents per day, what is the total cost of sickness to the wage-earners?

10. If compulsory health insurance were in operation according to a plan that the working man pays 25 cents a week, the employer 20 cents, and the government 5 cents, how much would each workman pay each year for health insurance? The employer? The government? (The benefits to be received, other methods of compensating workmen for sickness and other disabilities form the content of many related problems.)

Mathematics Preparation for Home Problems. In the foregoing groups of problems it is evident that some skill in computation, in quantitative thinking, and in mathematical analysis would be useful not only in the home but also in the daily lives of most self-supporting women who face many of the same situations as in the typical home. Several questions present themselves: Is the right kind of training being offered in the schools? What is the right kind of training? When and by whom should it be given? Is the interval between school and home-making (or independent living) too long to allow for effective preparation in school? Would a short "unit" course in connection with the required home economics course be valuable? Is there any "transfer of training" from courses such as algebra and geometry to situations in daily life where the methods of reasoning supposed to be developed in these courses would be useful? If not, could these courses be taught in such a way as to develop the power to apply mathematical analysis to practical situations?

For the answer to these and many similar questions further investigation and experimentation are necessary. If specific skills are desirable, if the capacity to analyze the pressing problems of living within a small income can be developed by mathematics courses, these facts are important to the makers of curricula. In view of the widespread dissatisfaction with the traditional courses in algebra and geometry and the tendency to allow students to graduate from high school without any knowledge of either, and even to enter college without these time-honored units it is highly important to consider whether or not the schools should offer to girls the special training in the mathematics they may need to make their contribution to the home life of this industrial age more effective.*

* Many of the problems have been adapted from "Household Arts Arithmetic" by Ball (Palmer) and West. The information required for the solution is found in supplementary tables in the same book.

Logarithms or Exponents?

By T. MUNRO
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ONE OF THE differences between high school mathematics as taught ten or fifteen years ago and mathematics as we strive to teach it today is a change of emphasis from the development of mere mechanical skills to a more thorough understanding of the fundamental processes involved in mathematical operations. As evidence of this fact one can point to recent changes in syllabi in which less emphasis is placed on such mechanical operations as factoring, involved quadratics, and so-called "apartment-house" fractions, and the introduction of trigonometry, the function concept, and extended work in graphs in the course in algebra.

While I was a graduate student at Teachers College, a professor remarked that he did not believe in teaching logarithms as such, but preferred to teach logarithmic work purely as an extension of the operations with exponents. I later studied the implications contained in this suggestion and put the idea into practice. The pages which follow give an account of the experiment and also show how the exponential and logarithmic methods compare when lined up side by side. Throughout, it is to be remembered that the chief purpose is to stress the *principle*, the underlying theory, in logarithmic work, notwithstanding the fact that the actual labor involved might be lengthened rather than shortened.

It is assumed at the outset that all teachers precede logarithmic work with a development of the fundamental principles underlying the use of exponents, such as:

1) $a^m \times a^n = a^{m+n}$	as $a^3 \times a^2 = a^5$
2) $a^m \div a^n = a^{m-n}$	as $a^5 \div a = a^4$
3) $(a^m)^n = a^{mn}$	as $(a^2)^3 = a^6$
4) $\sqrt[m]{a^n} = a^{n/m}$	as $\sqrt[3]{a^2} = a^{2/3}$

Using these four principles it is customary to develop the four laws for using logarithms, as: "In order to multiply, add logarithms," "in order to divide, subtract logarithms," etc. I believe that we de-

velop these "rules for using logarithms" (as they are so often captioned) too early and leave the exponents too much in the background.

In a recent experiment I conducted while teaching logarithms the term "logarithm" was never mentioned, "exponent" being used entirely in its place. Perhaps a few illustrations will best show how this was done. For purposes of comparison I have placed in parallel columns: first, the logarithmic method of attack which is perhaps the standard or most widely used method. Second I have shown the work as it was done in this experiment and which I shall call the "exponential method."

Illustrative Problem 1. By use of the formula $A = P(1+i)^n$ find the compound amount of \$900 drawing 4% interest compounded quarterly for 15 years.

Logarithmic Method	Exponential Method
(1) $A = P(1+i)^n$	$A = P(1+i)^n$ (1)
(2) $A = 900 \times 1.01^{60}$	$A = 900 \times 1.01^{60}$ (2)
(3) $\log A = \log 900 + 60 \log 1.01$	$A = 10^{2.954243} \times (10^{0.004321})^{60}$ (3)
(4) $\log A = 2.954243 + 60 \times 0.004321$	From the fundamental principle above, we have:
(5) $\log A = 2.954243 + 0.259260$	$A = 10^{2.954243} \times 10^{0.259260}$ (4)
	From the fundamental principle (1) above, we have:
(6) $\log A = 3.213503$	$A = 10^{3.213503}$ (5)
(7) $A =$ antilog 3.213503	
(8) $A =$ \$1634.90	$A =$ \$1634.90 (6)

If the fundamental principles underlying the use of exponents have been properly taught, the student should have little difficulty in proceeding from one step to the next. It should be noted that the "logarithmic equation" has been entirely removed from the process and no new principles have been introduced. From his previous work the student realizes fully that 0.004321 must be multiplied by 60 in proceeding to step (4) in the right-hand column. In making the corresponding step using the logarithmic method, I believe it is done more often by rule than by reason.

Illustrative Problem 2. Find to the nearest tenth of an inch the edge of a cube having a volume of 72 cubic inches.

$V = e^3$
$e = \sqrt[3]{V}$
$e = \sqrt[3]{72}$
$\log e = 1/3 \log 72$
$\log e = 1/3 \times 1.857332$
$\log e = 0.619111$
$e =$ antilog 0.619111
$e = 4.1601$
$e = 4.2$ inches

$V = e^3$
$e = \sqrt[3]{V}$
$e = \sqrt[3]{72}$
$e = \sqrt[3]{10^{1.857332}}$
$e = 10^{1.857332/3}$
$e = 10^{0.619111}$
$= 4.1601$
$e = 4.2$ inches

Illustrative Problem 3. If money is worth 6% annually find the amount of an annuity of \$250 deposited at the end of each year for 15 years.

This problem might be analyzed as follows:

\$250 deposited at end of first year will amount to $[A = P(1 + i)^n]$, $A = 250(1.06)^{14}$

\$250 deposited at end of second year will amount to $A = 250(1.06)^{13}$

. . . . . .

. . . . . .

\$250 deposited at end of 13th year will amount to $A = 250(1.06)^2$

\$250 deposited at end of 14th year will amount to $A = 250(1.06)$

\$250 deposited at end of 15th year will amount to $A = 250$

The sum may be represented:

$S = 250\ (1.06^{14} + 1.06^{13} + - - - - - 1.06^2 + 1.05 +)$

The quantity in the parentheses is a geometric series:

$1 + 1.06 + 1.06^2 - - - - - - 1.06^{14}$

$$S = \frac{ar^n - a}{r - 1} = \frac{1 \times 1.06^{15} - 1}{1.06 - 1} = \frac{1.06^{15}\ 1 - 1}{.06}$$

At this point pupils often attempt to carry out the entire solution, including subtraction, by means of logarithms. This will never be done if exponents are used.

$$S = \frac{(10^{0.025306})^{15} - 1}{.06} = \frac{10^{0.379590} - 1}{.06}$$

$$S = \frac{2.3966 - 1}{.06} = \frac{1.3966}{.06} = 23.277$$

$S = 23.277$

$S = \$250 \times 23.277 = \$5{,}819.25$

"In what length of time will a deposit of $600 amount to $1,000 in a savings bank that pays 2% every six months?"[1]

$$A = P(1 + i)^n$$
$$1{,}000 = \$600(1.02)^n$$
$$\frac{1{,}000}{600} = (1.02)^n$$
$$\frac{10^{3.000000}}{10^{2.778151}} = (10^{0.008600})^n$$
$$10^{0.221849} = 10^{0.0086n}$$

At this point it is easy for the pupil to see that if these two expressions are to have the same value, and the same base, 10, they must have equivalent exponents. In other words:

$$0.221849 = 0.0086n$$
$$\text{or, } n = \frac{0.221849}{0.0086} = 25.7 \text{ or } 13 \text{ years.}$$

It should be noticed that throughout the illustrative problems, the essential laws of exponents are brought into play as a logical outgrowth of the students previous work with integral exponents. It means that the logarithmic work is simply an extension of the work with exponents, an easy carry-over being accomplished from the field of positive integral exponents to that of fractional and negative exponents.

The main feature of this method, however, is the fact that the pupil can never forget that a logarithm is an exponent. If it is correct to assume that one of the modern trends in the teaching of mathematics is a shifting of emphasis from the mere mastery of skills, to a more thorough understanding of the underlying principles, I feel that to teach logarithms in the method described in this paper is a step in the right direction.

In conclusion, a word might be appropriate with regard to the question: "Will you teach logarithms wholly by the method of exponents, or will there be a shifting over to the traditional or standard form, with the accompanying use of the logarithmic equation, after it is felt that the student has gained a thorough understanding of the underlying principles?" I think that the answer to this question

[1] From *Commercial Algebra*, Book II, by Wentworth, Smith & Schlauch. Ginn and Co.

must depend upon the needs of individual pupils. If it is felt that the student is to continue his mathematics in college and perhaps in allied fields of science and research, it can be seen that there are good reasons for having him familiar with the conventional manner of handling logarithms. If, on the other hand, the members of the group involved are not expected to continue their work in mathematics, I believe it best to let the work stand, entirely in exponential form, rather than introduce the logarithmic style when there is nothing to be gained by so doing.

No effort has been made to "test" this method of teaching, or to compare "results" of equivalent groups taught by these two methods. If the reader is interested and cares to try this scheme as a teaching device, I should be most anxious to compare results further. It has been tried by another teacher at the High School of Commerce, and he joins me in saying that it has been a worth-while experiment, and that its results are encouraging and enlightening.

News Notes

The following program was presented at the Mathematics Section of the Ohio State Educational Conference of which Professor R. L. Morton of the University of Ohio is chairman:

"Trends toward individualism in mathematical instruction." R. L. Wilbur, Cincinnati Public Schools.

"What mathematics shall we teach? To whom and why?" Marie Gugle, Assistant Superintendent, Columbus.

"The compatibility of mind and mathematics." John P. Everett, President of the National Council of Teachers of Mathematics.

The Mathematics Section of the Western District of the Pennsylvania State Education Association met at Pittsburgh on April 11, 1931. At this meeting, the chairman, Dr. Elizabeth B. Cowley, gave a report on the Detroit meeting, and announced that the proposed affiliation with the National Council of Teachers of Mathematics had now been completed. The program of the morning was as follows:

"Creative work in high school mathematics," Vera Sanford, School of Education, Western Reserve University, Cleveland.

Discussion: Harry E. Long, Westinghouse High School, Pittsburgh. Charles F. W. McCready, South Hills High School, Pittsburgh.

Marin Mersenne

1588-1648

FATHER MERSENNE* is noteworthy not only for his writings on mathematics, astronomy, and music, but also for his services in spreading the ideas of other workers and in acting as a liaison between them. His period was one of great activity in mathematics and in science, but at that time there were few if any organizations to act as clearing houses of mathematical information. Through correspondence with Descartes and with other prominent scholars, and through a series of informal meetings which he sponsored in Paris, Mersenne served in a capacity later filled by the Royal Society, the French Academy of Sciences, and the other scientific organizations. In fact, the French Academy may be said to have developed from Mersenne's conferences. It was at one of these that Blaise Pascal presented his "Essay pour les Coniques."[1]

Like Fermat, Mersenne was interested in the theory of numbers. Euclid (IX, 36)[2] had shown that numbers which we would write in the form $2^{p-1}(2^p-1)$ are perfect, i.e., the number is equal to the sum of its factors, if the expression (2^p-1) is a prime number. Mersenne listed values of p which he said comprised all those below 257 for which this is true. Of these "Mersenne numbers," however, certain ones are listed by Professor Cajori as being composite.[3]

Mersenne was also instrumental in making Galileo's investigations known in France through a translation of Galileo's work published in 1634, two years after this work had been condemned by the Inquisition. The fact that Mersenne was himself an ecclesiastic makes this the more significant.

* Mersenne was a member of the Minimite order founded by St. François de Paule and not a Franciscan as Ball erroneously states.

[1] See David Eugene Smith, *Source Book in Mathematics*, New York, 1929. It should be noted that Pascal was then only sixteen years old and that the people who first saw this paper supposed it to have been written by Pascal's father.

[2] Sir Thomas L. Heath's *Thirteen Books of Euclid's Elements* should be consulted for Euclid's statement of this theorem.

[3] Florian Cajori, *History of Mathematics*, 1919, p. 167.

Report of the Second Committee on Geometry

By CHARLES M. AUSTIN
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History of the appointment of this committee

THE COLLEGE ENTRANCE EXAMINATION BOARD has been giving an examination in a combined unit of plane and solid geometry designated as Mathematics cd. Because of the small number of applicants, the College Board proposed dropping this examination. The action was delayed at the request of Professors Tyler and Beatley of Harvard pending an investigation to be undertaken by the Mathematical Association of America to determine the desirability and feasibility of including some solid geometry in the first course in geometry.

President Young of the Mathematical Association appointed a committee in December 1928 with Professor Jackson as chairman to pursue the investigation. Professor Jackson presented the matter to the National Council at its meeting in Cleveland in February 1929 and proposed that five members of the Council be added to his committee. This was done at the meeting of the directors on Friday, Feb. 22, 1929.

In December 1929 Professor Jackson sought to clarify the project by a statement which appears on Pages 487-88 of THE MATHEMATICS TEACHER, Volume 22, December 1929.

He stated that "the proposal . . . has been made that the College Board modify its requirements so as to bring about the more extensive introduction of courses including an appreciable amount of solid geometry in the first year of geometry in place of a part of the plane geometry ordinarily taught."

"The . . . committee is not to draw up detailed recommendations . . . but to discover whether there is sufficient interest in the project . . . to justify the board in proceeding. . ."

A report was presented by Professor Jackson at the Council meeting in 1930 in which he specifically confined himself to the agenda stated above. Chiefly, he reported expressions of opinion from persons engaged in teaching mathematics in college. There were some ex-

pressions of opinion from others engaged in teaching high school mathematics. There was evidence of interest in the project; as to the desirability of the project, the judgment of those quoted was about equally distributed for and against it.

In the discussion at the meeting of the Council in 1930 there was a distinct feeling that action for or against the project was premature until something definite was proposed. It was felt that there should be a clear statement whether the project involved the elimination of the third semester course in geometry; it was not at all clear that pedagogical, administrative or practical considerations involved in the proposed project had been considered. A resolution was passed calling for a second report in which some of these details be considered. President Everett appointed a new committee to make the report called for by this resolution.

Summary of opinions from persons who have had experience with the project

Miss Mary Kelly, Witchita, Kansas. (Letter to C. M. Austin, Feb. 10, 1931)

We have taught a course in plane and solid geometry in the tenth grade of Wichita High School for the past two years.

The space concept can be taught to these students many of whom will not take the regular course in solid geometry offered in the junior year. Teaching the combined course adds greater interest to the study of geometry.

The only difficulty is to know what work to omit in plane geometry to bring in the space concepts and applications of plane geometry in the solid without overcrowding the course.

Miss Sabin, Denver. (Letter to C. M. Austin, Nov. 23, 1930, and to W. W. Hart, Jan. 7, 1931)

We have given some solid geometry in the 10th grade following the College Board outline . . . elective course in solid geometry in 12th grade. Able students get solid geometry in 10th grade very well. Since this year . . . we have adopted a plane geometry text for grade 10 . . . most of the teachers will omit the greater part of the solid geometry which comes in the second part of the course, a few will incidentally give ideas on space . . . all through the course in plane geometry.

J. B. Orleans, Washington High School, New York City. (Letter to C. W. Austin, Dec. 11, 1930)

During past three years . . . one of my teachers and I have been trying to find how much of space intuition and mensurational part of solid geometry can be introduced into plane geometry . . . conclusion that combination of plane

and solid geometry . . . out of question; secondly, that . . . book six would have to be treated informally; third, . . . to make room . . . and not sacrifice too much plane geometry, the work must be started in the junior high school. If . . . informal geometry in junior high school and present some demonstrative geometry in 9, then plane and solid geometry . . . in 10 . . . I am in sympathy with the course.

J. P. McCormack, Secretary, Association of Chairman of Depts. of Mathematics in High Schools of New York City. (Letter to C. M. Austin, Oct. 25, 1930)

Purpose . . . develop space imagination . . . suggest that most . . . solid geometry be inductive . . . limited to imagination questions . . . number of book six theorems . . . too difficult for a pupil early in a course but . . . inductively would fit in nicely with plane geometry . . . might prove some theorems on the sphere, although visualizing would be sufficient . . . throughout books two and three . . . some facts could be proved . . . we might include volumes inductively or by means of Cavalieri's Theorem . . . area problems in book four . . . made by using surface of solids . . . would not add to the difficulties . . . if mostly inductive . . . it would take time . . . cutting would come up . . . with these omissions . . . plenty of time for inductive work in solid geometry and even for some proofs.

In letter of Nov. 8.

. . . Our people feel that solid geometry should be brought in whenever it fits the plane . . . that solid geometry should be largely . . . imagination . . . extending plane to solid geometry by analogy and illustration . . . no course of the kind we are contemplating given here . . . except in Horace Mann School . . . we were agreed that proofs could not successfully be undertaken by pupils in the public schools who did not have unit of plane geometry in the ninth grade.

(Letter to C. M. Austin, Dec. 16, 1930)

. . . our association . . . discussed at length . . . not yet ready to approve or disapprove any syllabus on combined course . . . unanimous . . . no reasonable possibility of putting into our schools any course which required the proofs of the solid geometry theorems in any way comparable to what would be done in the solid geometry course . . . no more than use the solid geometry as imagination . . . general feeling that we should go slow about recommending any course combining plane and solid geometry.

J. O. Hassler. (Letter to C. M. Austin, Nov. 19, 1930)

I was set to try it . . . but found so much of value in plane geometry that I had not enough time to enter solid geometry in the 8½ months' term of the University High School here. I sometimes wonder if I am in favor of it.

M. J. Newell, Evanston, Ill. (Letter to W. W. Hart, Jan. 10, 1931)

. . . In 1900 found both plane and solid geometry in third year high school here. . . . had sections . . . never feeling . . . that these subjects were not well

covered; . . . but . . . algebra had weeded out . . . students not mathematically minded or willing to pay the price in hard work. . . . Were compelled in 1910 to change to plane geometry in second year and algebra in third (to keep in step with other schools).

C. G. Leonard, Detroit. (Letter to C. M. Austin, Nov. 7, 1930)

. . . have a combined course in plane and solid geometry published as Volume Two of *Technical Mathematics* (John Wyley Co).

Miss Utley reports that ". . . this book is used only at the Cass Technical High School in Detroit."

G. H. Crandall, Culver Military Academy. (Letter to C. M. Austin, Dec. 3, 1930)

. . . for about five years we gave plane and solid geometry to a small group which was selected on the basis of an excellent record in all their work the previous year . . . cover all the work usually given in one and one-half years . . . the class moved more rapidly than normal . . . because of the careful selection. [The subject matter was taught in parallel arrangement.] Instead of fusion, this was more of an interpolation.

Miss May Wilt, University High School, W.Va. (Letter to C. M. Austin, Dec. 12, 1930)

Have been experimenting with . . . course in plane and solid geometry . . . no fixed syllabus.

L. W. Lavengood, Central High School, Tulsa, Okla. (Letter to W. W. Hart, Dec. 29, 1930)

. . . have made an effort to combine plane and solid geometry . . . fusing it . . . introduced intuitively in the first semester . . . to a selected group . . . present experience favorable . . . but may change our mind at end of year.

J. W. Foust, State Teachers College, Mt. Pleasant, Mich.

Corresponding with representative people in preparation for a talk on this subject in Michigan quoted one man as feeling that the ". . . movement . . . seemed premature," another that ". . . the ideas were tried at Kansas City, Mo. at one time and abandoned," that "the advocates of the plan proposed to use . . . solid geometry without proof." One teacher reported that "it is not intended to give the usual proofs of solid geometry," another that "the three dimensional material will have to be based on intuition and find expression in exercises." Miss Gertrude Allen, one of the exponents of the one year course has an extensive article in the Fifth Yearbook. She has been experimenting with this course for several years and strongly favors its use in the tenth grade.

W. W. Hart reports that prior to about 20 years ago combined courses in plane and solid geometry were very general throughout the

whole state of Wisconsin, where it was taught chiefly to pupils in the third or fourth year of the high school, following a textbook written by Mr. Shutts of the Whitewater Normal School, published by Atkinson, Mentzer, and Grover; that this text presented a tandem course embracing just about all of the theorems which are now in the list of the College Board; that the book followed a "suggestive method," by which was meant that the declarative statements of the old Wentworth text were replaced in the main by questions or suggestions which would lead an able pupil to secure the demonstrations. This course has been entirely abandoned in the state of Wisconsin.

These comments were solicited because the persons quoted had been referred to as having experience with the proposed combined course. These comments themselves are evidence only of interest in a combined course; they certainly are not evidence for the "desirability or feasibility" of such courses.

Opinions from persons who may not have tried the combined course

At the Milwaukee meeting of the Central Association of Science and Mathematics Teachers on November 25, 1930, Mr. C. A. Stone of the University High School of Chicago reported the results secured by a questionnaire sent to about 150 teachers.

One hundred and four favored instruction in solid geometry in High School; six opposed. Twenty-seven favored a one year course in plane and solid geometry; eighty-eight opposed it. Twenty-seven favored a fused course, eighteen, a tandem or parallel organization.

Mr. Seymour, Mathematics Supervisor of New York State recently sent out a questionnaire to 160 teachers. He received 131 replies. Twenty-six favored a combined course, 92 definitely opposed the combined course. The other 13 had various qualifications in their answers.

Reasons for a one year combined course in Plane and Solid Geometry

1. A combined course would give all pupils some conscious knowledge of three dimensional space. At the present time only about 10 per cent of the pupils studying plane geometry ever do any work in solid. Since we live in a world of three dimensions, this seems very unfortunate, to say the least.

2. A combined course would be more interesting. We live on a spherical globe and we are surrounded by objects having three dimensions. Thus, there is a richer field from which to take illustrations. The pupils would then be aware

of the three dimensions. This multitude of objects and illustrations would provide the student with a greater motive for the study of geometry.

3. All geometry is a single subject and should not be separated into parts. The combined course would avoid the criticism—just or unjust—of dividing the subject into two water tight compartments with no relations existing between the two.

4. Many parts of geometry were worked out by mature minds for mature minds. The tenth year course should include the simple fundamental portions of both plane and solid. The more difficult and complex parts could well be placed in a third semester course elective for fourth year pupils.

5. In plane geometry many definitions and concepts are necessarily limited. Thus, the word angle would take on a wider significance if all kinds of angles—plane, dihedral, trihedral, etc.—were considered simultaneously. This would be easy, because angles between planes are more in evidence than angles between lines. The same situation is found when we consider the topics of parallels, circles and loci.

6. The formulas of solid geometry should form a part of our general knowledge. Since only about 10 per cent of our pupils study solid geometry in the high school this condition will never be brought about by our present plan. The development of these formulas and the solution of a body of sensible, significant and interesting problems would make a lasting impression on the tenth grade pupils.

7. The third semester course referred to above might also include some elementary notions of the calculus. This plan would gain time for the student by bringing him more quickly to this most useful of the mathematical studies.

Reasons against a combined course in plane and solid geometry

1. Tenth year pupils are not mature enough to study solid geometry. Indeed it seems next to impossible for them to comprehend some of the principles of plane geometry. They seem unable to visualize drawings in one plane. What would they do with drawings in more than one plane?

2. The experimental basis for this course is not yet strong enough. The committee found only one school system where such a course was in use. Occasional schools and single classes in experimental schools are trying out the plan. In the College Entrance Board examination in June 1930 only six candidates took the test in Mathematics cd.

3. Sufficient intuitive geometry is not given in our elementary schools for the foundation of such a course. European schools spread out their work in geometry over a period of years, some beginning as early as the fourth grade.

4. Our present plane geometry course would be wrecked. It has been cut in so many places already that no more operations can be tolerated. At the same time, sufficient solid geometry material could not be introduced to insure any mastery in that field.

5. Our high schools are now full of pupils in the lower levels of ability. They have great difficulty in completing our present course, even in its denatured state, in one year. In some schools these slower pupils are required to spend

three semesters on plane geometry. What would they do if we bring in the more difficult concepts of solid?

6. The present course in solid geometry is the best course for maturing the pupils mathematically. They appreciate the close logical reasoning and the selection of just the proper words to convey a definite meaning. A superficial course in the tenth year would take the zest out of any following course.

7. It is assumed here that the object of geometry teaching is the cultivation of postulational thinking. Now, it seems to be the opinion of many teachers that a combined course would seriously interfere with the realization of this objective. If so, then the combined course is wrong.

8. Too many assumptions necessary. It is wrong to assume difficult theorems in order to prove a few easy originals.

9. Administrative difficulties bar the way for such a course. New and undefined courses work a hardship upon pupils who transfer from one school to another.

10. Suitable texts or syllabi are not available. Extemporaneous teaching even by experienced teachers, is not advisable.

Suggestions from the Committee

The committee herewith presents two tentative courses combining plane and solid geometry.

The first is called a tandem course because the material from solid geometry is introduced separately after the plane is finished. In this course proofs are required for fifty-six theorems of plane geometry and twenty-eight of solid. Definitions and explanatory material would have to be given at the proper places. Also, corollaries, needed to complete the sequence and to introduce other principles would have to be included. In addition, exercises would be needed to emphasize and elucidate important theorems. It is through these original exercises, solved by analysis, that the student obtains the real benefit from his study of geometry.

The second is called a fused course. In this course, the solid geometry concepts are introduced at appropriate places by imagination questions. Rigid proofs are required only for the plane geometry theorems. The answers to the imagination questions are, however, not obvious. The pupil must visualize each situation and think in order to obtain the correct answer. In this course thirty-eight theorems and eighteen corollaries of plane geometry are proved. The usual construction problems and numerous original exercises would be provided. A thorough study of proof by analysis would be included in the course. Besides the imagination questions of solid geometry, proofs for fifteen theorems are required. Most of the formulas are assumed

or developed intuitively. Many numerical exercises depending on these formulas would be included to complete the course.

Since our meeting in Atlantic City (1930), two outlines for one-year courses have been published in THE MATHEMATICS TEACHER. One, a tandem course, was proposed by George W. Evans in the issue of February 1930. His course contains 104 propositions—53 of plane geometry and 51 of solid. The other by W. D. Reeve and his students at Columbia appeared in the issue of October, 1930. This is a fused course, the material from solid geometry being introduced along with the analogous plane geometry. It includes seventy-one theorems and sixteen constructions from plane geometry and eleven theorems from solid. The formulas of solid geometry are to be developed intuitively. The authors of this outline would include numerous original exercises to be solved by analysis.*

The syllabus for Mathematics cd of the College Entrance Examination Board includes ninety-eight theorems and nineteen constructions.

The course used in Wichita, Kansas is a fused course. Their printed syllabus tells what concepts of solid geometry to use. It also gives specific directions to the teacher, telling where and how these concepts shall be presented.

Here then are six different courses from which one may choose. The two proposed by the committee are shorter and simpler than the others.

The committee hopes that these courses may be helpful to teachers who are experimenting with this problem.

Let it be distinctly understood that the committee is not recommending these courses for adoption. They are tentative only, and if used should be for the purpose of experimentation. The final word can be said only by those who actually make a trial of these things in the classroom. Mr. Hart says that by actual trial, tenth year pupils of good ability can master the theorems of solid geometry. Granting that, the question then remains, is it advisable to upset our present course in plane geometry in order that all pupils may have the opportunity to study in some small, superficial way the geometry of three dimensions?

If a one-year course is used, at least two members of the committee

* To this should be added the outline suggested by Professor Ralph Beatley of Harvard, appearing in the issue of April, 1931.—Editor.

feel that there should be a third semester of geometry offered as an elective. Third semester course would include a thorough discussion of the theory of limits and other difficult portions of both plane and solid geometry.

Our study of this question brings the following conclusions:

1. Although believing that it would be valuable for all students to have a knowledge of three dimensional geometry, the committee is not ready to recommend the adoption of a one-year course in plane and solid gometry for all tenth year pupils.

2. Practically all the people who have experimented with this problem and who favor the combined course agree that the work in solid geometry should be done intuitively.

3. It does not seem possible to do this combined course in the tenth year unless it is preceded by some intuitive geometry of a very high grade, both in plane and solid, in the Junior High School.

4. At present the experimental basis is not strong enough to warrant the institution of the one-year course. Although tried in several places, the course has not been continued.

5. If by experimentation a satisfactory one-year course in geometry can be arranged, it should be followed by a third semester course containing the more difficult parts of both plane and solid. This course would be taken only by those who desire a complete knowledge of geometry and a better preparation for their advanced work.

6. Under present conditions about the best thing teachers can do is to introduce into our plane geometry course the analogous solid geometry. Thus, when studying areas in Book IV, the lateral and total areas of the prism, pyramid and frustum of pyramid could easily be deduced. Then in Book V, might follow a study of cylinder and cone.

7. Other excursions into solid geometry might be made when studying angles, parallels, circles and loci.

8. Even this work is not advised without careful preparation by the teacher, both as to material and to method.

Syllabus of a Proposed One-Year Course in Geometry

Book I

Assumptions

1. A straight line may be drawn between any two points.
2. A straight line may be extended as far as desired.
3. A circle may be drawn with any point as a center and any radius.
4. Quantities equal to the same quantity, or equal quantities, are equal to each other.
5. If equals are added to equals, the sums are equal.
6. If equals are subtracted from equals, the remainders are equal.

7. If equals are multiplied by equals, the products are equal.

8. If equals are divided by equals, the quotients are equal.

9. In any operation, a quantity may be substituted for its equal.

10. A whole is equal to the sum of all of its parts and is greater than any one of its parts.

11. Only one straight line can be drawn between two points.

12. Two straight lines can intersect in only one point.

13. A straight line is the shortest distance between two points.

14. Through a given point not more than one line can be drawn parallel to a given line.

15. Through a given point only one perpendicular can be drawn to a given line.

16. All straight angles are equal.

17. All right angles are equal.

18. The sum of all the angles about a point equals two straight angles or 360 degrees.

19. The sum of all the angles about a point on one side of a straight line equals one straight angle or 180 degrees.

20. The complements of equal angles are equal.

21. The supplements of equal angles are equal.

22. Two triangles are congruent, if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.

23. Two triangles are congruent, if two angles and the included side of one are equal respectively to two angles and the included side of the other.

24. Two triangles are congruent, if three sides of one are equal to three sides of the other.

25. Two right triangles are congruent if the hypotenuse and a side of one are equal respectively to the hypotenuse and a side of the other.

Theorems

1. If two straight lines intersect the vertical angles are equal.

2. If two sides of a triangle are equal the angles opposite these sides are equal.

3. Two lines perpendicular to the same line are parallel.

4. Two right triangles are congruent if the legs of one are equal respectively to the legs of the other.

5. Two right triangles are congruent, if the hypotenuse and an adjacent angle of one are equal respectively to the hypotenuse and the corresponding adjacent angle of the other.

6. If two parallel lines are cut by a transversal the alternate interior angles are equal.

7. If two lines are cut by a transversal so that the alternate interior angles are equal, the lines are parallel.

8. If two parallel lines are cut by a transversal, the corresponding angles are equal.

9. If two lines are cut by a transversal so that the corresponding angles are equal, the lines are parallel.

10. The sum of the three angles of a triangle equals one straight angle.

11. If two angles have their sides respectively parallel or perpendicular, they are either equal or supplementary.

12. If two angles of a triangle are equal the sides opposite these angles are equal and the triangle is isosceles.

13. All points on the perpendicular bisector of a line are equally distant from the extremities of the line; and conversely, all points equally distant from the extremities of a line lie on the perpendicular bisector of the line.

14. All points on the bisector of an angle are equally distant from the sides of the angle; and converesly, all points equally distant from the sides of an angle lie on the bisector of the angle.

15. The opposite sides and angles of a parallelogram are equal.

16. If the opposite sides of a quadrilateral are equal, it is a parallelogram.

17. If two sides of a quadrilateral are equal and parallel, it is a parallelogram.

18. If three or more parallels intercept equal segments on one transversal, they intercept equal segments on every transversal.

Constructions

1. Bisector of line.
2. Angle equal to a given angle.
3. Perpendicular to given line from point on the line.
4. Perpendicular to given line from point outside the line.
5. Bisector of angle.
6. Line through given point parallel to given line.
7. Divide a line into any number of equal parts.
8. Triangle when three sides are given.
9. Triangle when two sides and included angle are given.
10. Triangle when two angles and included side are given.

BOOK II

Assumptions

1. All diameters of the same circle or of equal circles are equal.
2. All radii of the same circle or of equal circles are equal.
3. A central angle is equal in degrees to its intercepted arc.
4. In the same circle or in equal circles, equal central angles intercept equal arcs.
5. In the same circle or in equal circles equal arcs subtend equal central angles.

Theorems

1. In the same circle or in equal circles, two central angles are proportional to their intercepted arcs.

2. In the same circle or in equal circles equal chords subtend equal arcs; and conversely, equal arcs are subtended by equal chords.

3. In the same circle or in equal circles, equal chords are equidistant from the center; and conversely, chords equidistant from the center are equal.

4. A diameter perpendicular to a chord of a circle bisects the chord and the arcs subtended by it.

5. A line perpendicular to a radius at its outer extremity is tangent to the circle.

6. A line tangent to a circle is perpendicular to the radius drawn to the point of tangency.

7. Two tangents drawn to a circle from the same point are equal.

8. Two parallel lines—chords, secants, tangents or one secant and a tangent—intercept equal arcs on a circle.

9. An inscribed angle is measured by one-half its intercepted arc.

10. An angle formed by two intersecting chords of a circle is measured by one-half the sum of the two intercepted arcs.

11. An angle formed by a chord and a tangent is measured by one-half its intercepted arc.

Constructions

1. Bisect arc.
2. Circle through any three points not in same straight line.
3. Tangent to a circle from external point.
4. Tangent to a circle from point on circle.
5. Circumscribe circle about a triangle.
6. Inscribe circle in triangle.

Book III

Theory of Proportion

1. In any proportion the product of the means equals the product of the extremes.

2. If the product of two numbers equals the product of two other numbers, one pair may be made means and the other pair the extremes of a proportion.

3. Any proportion may be written by alternation.

4. Any proportion may be written by inversion.

5. If three terms of one proportion equal respectively the three corresponding terms of another proportion, the fourth terms are equal.

Theorems

1. A line parallel to one side of a triangle and intersecting the other two sides divides these two sides proportionally.

2. A line that divides two sides of a triangle proportionally is parallel to the third side.

3. Two triangles are similar if the angles of one are equal respectively to the angles of the other.

4. Two triangles are similar if an angle of one equals an angle of the other and the sides including these angles are proportional.

5. Parallel lines cut off proportional segments on any two transversals.

6. In any right triangle, the perpendicular drawn from the vertex of the right angle to the hypotenuse divides the triangle into two triangles, each similar to the given triangle and to each other.

7. In any right triangle, the square of the hypotenuse is equal to the sum of the squares of the two legs.

8. If two chords intersect in a circle, the product of the segments of one chord equals the product of the segments of the other chord.

9. If a tangent and a secant are drawn to a circle from an external point, the tangent is a mean proportional between the secant and its external segment.

10. Two similar polygons can be separated into the same number of triangles similar each to each and similarly placed.

11. In a series of equal fractions, the sum of the numerators divided by the sum of the denominators equals any one of the fractions.

12. The perimeters of any two similar polygons are proportional to any two corresponding sides.

Constructions

1. Divide a line proportionally.
2. A mean proportional between any two line segments.
3. A fourth proportional to three given line segments.
4. A polygon similar to a given polygon.

BOOK IV

Assumption

1. The area of a rectangle equals the product of its base and altitude.

Theorems

1. The area of a parallelogram equals the product of its base and altitude.
2. The area of a triangle equals one-half the product of its base and altitude.
3. The area of a trapezoid equals one-half the product of the altitude and the sum of the two bases.
4. The areas of two similar triangles are proportional to the squares of any two corresponding sides.
5. The areas of two similar polygons are proportional to the squares of any two corresponding sides.

BOOK V

Assumptions

1. If a regular polygon is inscribed in a circle and the number of its sides is indefinitely increased, (a) the perimeter of the polygon approaches the circumference of the circle as a limit; (b) the apothem of the polygon approaches the radius of the circle as a limit; (c) the area of the polygon approaches the area of the circle as a limit.

Any fact that is true of regular polygons and independent of the number of sides is also true of circles.

Theorems

1. A circle can be circumscribed about a regular polygon.
2. A circle can be inscribed in a regular polygon.
3. If a circle is divided into any number of equal arcs, the chords of these arcs form a regular inscribed polygon; and the tangents drawn at the points of division form a regular circumscribed polygon.

4. The area of a regular polygon equals one-half the product of its perimeter and apothem.

5. Regular polygons having the same number of sides are similar.

6. The perimeter of two regular polygons having the same number of sides are proportional to the radii of the inscribed or the radii of the circumscribed circle.

7. The circumferences of two circles are to each other as their radii.

8. The circumference of a circle equals $2\pi r$.

9. The area of a circle equals one-half the product of its circumference and radius.

Constructions

1. Inscribe a square in a circle.

2. Inscribe a regular hexagon in a circle.

BOOK VI

Assumptions

1. If two planes meet, their intersection is a straight line.

2. Through a given point in a line one plane and only one can be drawn perpendicular to the line.

3. Through a given external point one plane and only one can be drawn perpendicular to a given line.

4. The perpendicular is the shortest line from a point to a plane.

5. Two line perpendicular to same plane are parallel.

6. Two lines parallel to a third line are parallel to each other.

7. If one of two parallel lines is perpendicular to a plane, the other is also perpendicular to the plane.

8. Two planes perpendicular to same line are parallel.

9. A line perpendicular to one of two parallel planes is perpendicular to the other.

10. Through a given point outside a plane one plane and only one can be drawn parallel to the given plane.

Theorems

1. Oblique lines drawn from a point to a plane and meeting the plane at equal distances from the foot of the perpendicular are equal.

2. If a line is perpendicular to each of two intersecting lines at their point of intersection, it is perpendicular to the plane of the lines.

3. If two lines are parallel, a plane containing one of the lines and only one, is parallel to the other.

4. If two intersecting lines are each parallel to a plane, the plane of these lines is parallel to the given plane.

5. If two angles, not in the same plane, have their sides parallel each to each and extending in the same direction from their vertices, the angles are equal.

6. If two lines are cut by three parallel planes, their corresponding intercepts are proportional.

7. All points equidistant from two given points lie in a plane perpendicular to the line joining them at its middle point; and

8. All points equidistant from the sides of an angle lie in the plane bisecting the angle:

9. Any point in the plane bisecting a dihedral angle is equidistant from the faces of the angle.

Book VII

Assumptions

1. Two right prisms having equal bases and equal altitudes are equal.

2. The plane drawn through two diagonally opposite edges of a right parallelopiped divides it into two equivalent triangular prisms.

3. The volume of a right parallelopiped equals the product of its base by its altitude.

4. A triangular prism can be divided into three equivalent triangular pyramids.

Theorems

1. The lateral area of a right prism equals the product of its altitude by the perimeter of the base.

2. The volume of a right triangular prism equals the products of its base and altitude.

3. The volume of any prism equals the product of its base and altitude.

4. If a pyramid is cut by a plane parallel to the base, the lateral edges and the altitude are divided proportionally and the section made is a polygon similar to the base.

5. The lateral area of a regular pyramid equals one half the product of the slant height by the perimeter of the base.

6. The lateral area of a frustum of a regular pyramid equals one-half the sum of the perimeters of the two bases multiplied by the slant height of the frustum.

7. The volume of a right triangular pyramid equals one-third the product of its base by its altitude.

8. The volume of any pyramid equals one-third the product of its base by its altitude.

Book VIII

Assumptions

1. If a right prism is inscribed in a right circular cylinder and the number of sides of its basis is indefinitely increased, (a) the lateral area of the prism approaches the lateral area of the cylinder as a limit; (b) the volume of the prism approaches the volume of the cylinder as a limit.

2. If a right pyramid is inscribed in a right circular cone and the number of sides of its base is indefinitely increased, (a) the lateral area of the pyramid approaches the lateral area of the come as a limit; (b) the volume of the pyramid approaches the volume of the cone as a limit.

Theorems

1. The bases of a cylinder are equal.
2. The lateral area of a right circular cylinder equals the product of the altitude by the circumference of the base.
3. The volume of a right circular cylinder equals the product of its base by its altitude.
4. The section of a circular cone made by a plane parallel to the base is a circle.
5. The lateral area of a right circular cone equals one-half the product of its slant height by the circumference of its base.
6. The lateral area of a frustum of a right circular cone equals one-half the sum of the circumferences of its base multiplied by the slant height.
7. The volume of a right circular cone equals one-third the product of the base by its altitude.

BOOK IX

Theorems

1. If a plane is passed through a sphere, the intersection is a circle.
2. The area of the surface generated by a straight line revolving about an axis in its plane equals the product on the projection of the line upon the axis by the circumference of a circle whose radius is the line perpendicular to the revolving line at its mid-point and terminating in the axis.
3. The surface of a sphere equals $4\pi r^2$.
4. The volume of a sphere equals $\frac{4}{3}\pi r^3$.

A TENTATIVE COURSE COMBINING PLANE AND SOLID GEOMETRY

Developed by J. P. McCormack

AXIOMS AND POSTULATES

Axioms:

1. In any process a quantity may be substituted for an equal one.
2. Things equal to the same thing or to equal things are equal.
3. If equals are added to equals, the results are equal.
4. If equals are subtracted from equals, the results are equal.
5. If equals are multiplied by equals, the results are equal.
6. If equals are divided by equals, the results are equal.
7. The whole equals the sum of its parts.
8. The whole is greater than any of its parts.
9. A straight line is the shortest line between two points.
10. A geometric figure can be moved without change of size or shape.
11. Through a point not more than one line can be drawn parallel to a given line.
12. The perpendicular is the shortest line from a point to a line.
13. Every angle has a bisector.
14. A line segment has a middle point.

Postulates:

1. A straight line can be drawn between any two points.
2. A straight line can be extended as far as desired.
3. A circle or arc can be constructed with any center and any radius.

Assumed Theorems:

1. All straight angles are equal.
2. All right angles are equal.
3. Complements of the same angle or of equal angles are equal.
4. Supplements of the same angle of equal angles are equal.
5. If two adjacent angles have their exterior sides in a straight line, they are supplementary.
6. Vertical angles are equal.

Construction

These constructions should be taken largely for the purpose of clearing up the pupils' ideas of angle, perpendiculars, triangles, etc.

1. Construct the perpendicular bisector of a line segment.
2. Construct a perpendicular to a line through a given point.
3. Bisect a given angle.
4. Construct an angle equal to a given angle.
5. Use these constructions in exercises.
6. Construct a triangle congruent to a given triangle.
   (a) Using the three sides.
   (b) Using two sides and their included angle.
   (c) Using two angles and their included side.
7. Emphasize the results of these constructions and then postulate the corresponding theorems about congruent triangles.
8. Motivate by using applied problems such as finding the distance across a river.
9. Prove many exercises on congruent triangles.
10. Study the analytic method of proof.

THEOREM 1. *The base angles of an isosceles triangle are equal.*

Solid Geometry

1. Discuss the four ways of determining a plane. Illustrate by rotating a cardboard when one point and then two points are fixed.
2. Discuss the intersection of two planes. Illustrate by folding paper.
3. Take imagination exercises of the following types:
   (a) Can three planes intersect in just one line? In just two lines? etc.
   (b) How many planes are determined by four points not all in the same plane?
   (c) Does an arc of a circle determine a plane?

PERPENDICULARS AND PARALLELS

Plane Geometry

THEOREM 2. *Any point on the perpendicular bisector of a line segment is equally distant from the ends of the segments, and conversely.*

1. Take exercises on perpendicular bisectors.

2. Illustrate or prove as an exercise: An exterior angle of a triangle is larger than either remote interior angle. Vary the shape of the triangle and show that it is always true.

3. Define and discuss: Parallel lines; alternate interior angles; corresponding angles, etc.

THEOREM 3. *Two lines are parallel if their alternate interior angles are equal.*

COR. *Two lines are parallel if their corresponding angles are equal.*

COR. *Lines perpendicular to the same line are parallel.*

COR. *Not more than one perpendicular to a line can be drawn through a point.*

THEOREM 4. *Through a given point construct a line parallel to a given line.*

THEOREM 5. *Alternate interior angles of parallel lines are equal.*

COR. *A line perpendicular to one of two parallel lines is perpendicular to the other.*

COR. *Lines parallel to the same line are parallel.*

COR. *Angles having their sides respectively parallel are either equal or supplementary.*

4. Take many exercises on parallel lines.

Solid Geometry

1. Extend the ideas of parallel and perpendicular to two lines in space, two planes, and a line and a plane. Give the pupil the idea of perpendicular planes and of a line perpendicular to a plane by illustration rather than by exact definition. Bring in the idea of dihedral angles and their plane angles.

2. Take a large number of imagination questions to bring out ideas about these relations. Do not require proof, but illustrate by using cardboard and pencils, or other devices. The following exercises illustrate the type:

Imagination Exercises

1. Are two planes parallel if they are:
   (a) parallel to the same plane?
   (b) perpendicular to the same plane?
   (c) parallel to the same line?
   (d) perpendicular to the same line?
2. Are two lines parallel if they are:
   (a) parallel to the same line?
   (b) perpendicular to the same line?
   (c) parallel to the same plane?
   (d) perpendicular to the same plane?
3. Are two planes perpendicular if one of them:
   (a) contains a line perpendicular to the other?
   (b) is parallel to a line which is perpendicular to the other?

Questions of this type can be asked to bring out nearly all of the theorems of Book 6, and the answers are not obvious. The pupil must visualize.

3. Next attack the corollaries of theorems 3 and 5, to see if they still hold in three dimensions.

4. Prove (as an exercise, not a proposition): The intersections of two parallel planes with a third plane are parallel. Other propositions of solid geometry may be proved as exercises here.

TRIANGLE THEOREMS

Plane Geometry

THEOREM 6. *The sum of the angles of a triangle is a straight angle.*

COR. *Each angle of an equilateral triangle equals 60°.*

COR. *If two triangles have two angles of one equal respectively to the two angles of the other, their third angles are equal.*

COR. *Two triangles are congruent if two angles and a side opposite one of them equals two angles and the corresponding side of the other.*

1. Vary the triangle and show that the relation still holds when one angle becomes 0° or 180° or when the vertex moves off toward infinity.

THEOREM 7. *If two angles of a triangle are equal, the sides opposite them are equal.*

THEOREM 8. *Two right triangles are congruent if the hypotenuse and a leg of one equals the hypotenuse and a leg of the other.*

2. Take exercises on these theorems.

THEOREM 9. *Any point on the bisector of an angle is equally distant from the sides.*

Solid Geometry

1. Imagination exercises. Are theorems 6-9 true for a triangle on a sphere? In theorem 6, show that it is not true by drawing a trirectangular triangle on a ball.

2. Consider a point on the plane bisecting a dihedral angle.

PARALLELOGRAMS

Plane Geometry

1. Definitions of parallelograms, rectangles, rhombus, square.

THEOREM 10. *A diagonal of a parallelogram divides it into two congruent triangles.*

COR. *The opposite sides of a parallelogram are equal.*

THEOREM 11. *A quadrilateral is a parallelogram if its opposite sides are equal.*

THEOREM 12. *A quadrilateral is a parallelogram if two sides are equal and parallel.*

THEOREM 13. *If two or more parallel lines cut off equal lengths on one transversal, they cut off equal lengths on every transversal.*

2. Take exercises on parallelograms.

Solid Geometry

1. Definitions of polyhedron, tetrahedron, prism, parallelopiped.

2. Prove the following propositions as exercises, not as propositions to be remembered.

(a) The sections of a prism made by parallel planes cutting all the lateral edges are congruent polygons.

(b) The bases of a prism are congruent polygons.

(c) The opposite faces of a parallelopiped are congruent parallelograms.

3. Prove other simple exercises using solid figures, such as:

(a) All the lateral edges of a prism are equal.

(b) If a plane is passed through two lateral edges of a prism, the section is a parallelogram.

4. Imagination question such as:

Are the lateral faces of a prism parallel to the lateral edges not in the face?

CIRCLES

Plane Geometry

1. Define circle, radius, arc, chord, tangent, etc.

2. Illustrate and assume:

(a) Radii of a circle or of equal circles are equal.

(b) Circles are equal if their radii are equal.

(c) A point is inside, on or outside a circle according as its distance from the center is less than, equal to or greater than the radius.

(d) A central angle is measured by its arc.

COR. *In a circle, equal central angles intercept equal arcs, and conversely.*

COR. *In a circle, equal chords have equal arcs, and conversely.*

THEOREM 14. *A radius perpendicular to a chord bisects the chord and its arc.*

THEOREM 15. *In a circle, equal chords are equally distant from the center, and conversely.*

Function idea. Move a chord parallel to itself until it becomes a tangent. Take exercises on chords.

Solid Geometry

1. Define sphere, etc.

2. Prove as an exercise: A section of a sphere made by a plane is a circle.

3. Define great circle, small circle, axis. Call attention to the fact that on the sphere great circle arc takes the place of straight line.

4. Discuss great circles and their relation to navigation and aviation. Show why an airplane flying from New York to Paris will fly through points farther north than either terminal.

5. Discuss generating a sphere by rotating a circle about a diameter.

6. Prove that circles of a sphere equally distant from the center are equal.

TANGENTS

Plane Geometry

THEOREM 16. *A tangent is perpendicular to the radius to the point of contact, and conversely.*

COR. *Tangents to a circle from a point are equal.*

Solid Geometry

1. Take imagination questions such as:
   (a) Is a plane tangent to a sphere perpendicular to a radius to the point of contact?
2. Discuss the tangent cone from a point to a sphere. Obtain it by rotation of a circle and a tangent line.

MEASUREMENT OF ANGLES

Plane Geometry

THEOREM 17. *An inscribed angle is measured by half its intercepted arc.*

THEOREM 18. *An angle formed by a tangent and a chord is measured by half its intercepted arc.*

1. Prove as corollaries the measurement of the angle formed by intersecting chords and that outside the circle made by tangents or secants or both.
2. Function idea. Show that these theorems are special cases of one theorem by moving the lines or the circle.
3. Take exercises on measurement of angles.

Solid Geometry

Define the spherical angles and discuss its measurement.

CONSTRUCTIONS

Plane Geometry

THEOREM 19. *A circle can be circumscribed about a given triangle.*

THEOREM 20. *A circle can be inscribed in a given triangle.*

THEOREM 21. *Through a given point construct a tangent to a circle.*

1. Take geometric analysis of construction problems and apply exercises.

Solid Geometry

Discuss (or possibly prove):
   (a) A sphere can be circumscribed about any tetrahedron.
   (b) A sphere can be inscribed in any tetrahedron.
2. Imagination questions such as:
   (a) How many points determine a circle? A sphere?
3. Discuss theorem 19 and its solid analogue when the center moves to the boundary, and outside, then to infinity and back on the other side.

LOCI

Plane Geometry

1. Definition and discussion. Define the locus as of a moving point. The function idea is shown better this way than by the place of a series of fixed points.

2. Take the perpendicular bisector of a line segment and the bisector of an angle.

3. Take exercises on loci, first one locus, then two loci determining a point. Most of these are imagination questions without proof.

Solid Geometry

1. Extend the idea of a locus to three dimensions.

2. Discuss the plane perpendicular bisector of a line segment and the plane bisector of a dihedral angle.

3. Take imagination exercises on loci in space without proof. First one locus then two determining a line, and finally three determining a plane. Parallel the plane loci with the corresponding space loci for comparison.

Illustration: I am thinking of a point two feet from the front room. Where is it? It is also three feet from the side wall of the room. Where is it now? And it is also four feet from the floor. Show me where this point is.

PROPORTION AND SIMILAR TRIANGLES

Plane Geometry

1. A discussion of proportion with numerical exercises. Illustrate the following:

(a) In a proportion the product of the extremes equals the product of the means.

(b) If the product of two quantities equals the product of two others, one pair may be made the means and the other pair the extremes of a proportion.

(c) The mean proportional between two numbers is the square root of their product.

(d) If three terms of one proportion equal respectively the three corresponding terms of another proportion, their remaining terms are equal.

(e) In any proportion the terms are also in proportion by addition.

2. Discuss (or prove): A line parallel to one side of a triangle divides the other two sides into proportional segments.

THEOREM 22. *A line that divides two sides of a triangle proportionally is parallel to the third side.*

THEOREM 23. *Two triangles are similar if two angles of one equal respectively two angles of the other.*

COR. *Triangles similar to the same triangle are similar to each other.*

COR. *In similar triangles corresponding altitudes are to each other as corresponding sides.*

THEOREM 24. *Two triangles are similar if an angle of one equals an angle of the other and the including sides are proportional.*

COR. *A line bisecting two sides of a triangle is parallel to the third side and equal to half the third side.*

3. Take many exercises on similar triangles, and particularly applications.

Solid Geometry

1. Consider the possibility of similar triangles on a sphere; of parallel arcs of great circles. Use a ball for Illustration.
2. Define pyramid and cone and show models.
3. Prove as an exercise: If a pyramid is cut by a plane parallel to its base:
   (a) The lateral edges and altitude are divided proportionally.
   (b) The section is a polygon similar to the base.
4. Prove as an exercise: A section of a circular cone made by a plane parallel to the base is a circle.

RIGHT TRIANGLE THEOREMS

Plane Geometry

THEOREM 25. *In a right triangle, the altitude on the hypotenuse is the mean proportional between the segments of the hypotenuse, and each leg is a mean proportional between the hypotenuse and the segment adjacent to that leg.*

THEOREM 26. *In a right triangle, the square of the hypotenuse equals the sum of the squares of the legs.*

COR. TO 25. *A perpendicular to a diameter from any point on the circle is a mean proportional between the segments of the diameter.*

1. Exercises on the right triangle including many applications.

Solid Geometry

1. Exercises in which the right triangles are not in a single plane, such as finding the diagonal of a rectangular solid, the slant height of a regular pyramid.

CONSTRUCTIONS

Plane Geometry

THEOREM 27. *Construct the fourth proportional to three given line segments.*

THEOREM 28. *Divide a line segment into parts proportional to given line segments.*

COR. *Divide a line segment into any number of equal parts.*

THEOREM 29. *Construct the mean proportional between two given line segments.*

TRIGONOMETRY

Plane Geometry

1. Review the use of the sine and tangent in computation.
2. Optional use of the law of sines.
3. Take many applications of these functions.
4. Discussion of the sine and tangent as functions of a varying angle.

PROPORTIONAL LINES IN THE CIRCLE

Plane Geometry

THEOREM 30. *If two chords intersect, the product of the segments of one equals the product of the segments of the other.*

1. Discuss the functional relations obtained by moving the chords until they:
   (a) intersect on the circle.
   (b) intersect outside the circle.
   (c) one or both become tangents.
   (d) Prove these cases as exercises.
2. Take applications based on these.

Solid Geometry

1. Discuss the extension of (c) to the tangent line to a sphere, and find the distance which one can see from a given height at sea.

AREAS

Plane Geometry

1. Assume the area of the rectangle.

THEOREM 31. *The area of a parallelogram equals the product of its base and altitude.*

THEOREM 32 *The area of a triangle equals half the product of its base and altitude.*

THEOREM 33. *The area of a tropezoid equals half its altitude times the sum of its bases.*

2. By varying a base of the trapezoid obtain the areas of the rectangle, parallelogram and triangle as special cases.
3. Take many computational exercises on areas.

THEOREM 34. *The areas of similar triangles are to each other as the squares of corresponding sides.*

4. Take construction by algebraic analysis.
5. Take construction exercises.

Solid Geometry

1. Apply the above area theorems to the surfaces of prisms, parallelopipeds, pyramids, etc. Develop formulas for their lateral and total areas.
2. Assume, discuss and use formulas for the volume of prisms and pyramids.
3. Using the triangle and square, show the physical law of inverse squares as applied to light. Solve exercises.

REGULAR POLYGONS AND THE CIRCLE

Plane Geometry

THEOREM 35. *A circle can be circumscribed about any regular polygon.*

THEOREM 36. *The area of a regular polygon equals half the product of its perimeter and the radius of its inscribed circle.*

1. Assume and use the formulas for the circumference and area of a circle.
2. Take exercises using these formulas.

THEOREM 37. *A square can be inscribed in a circle.*

THEOREM 38. *A regular hexagon can be inscribed in a circle.*

Solid Geometry

1. Discuss the volumes of the cylinder, cone and sphere and assume formulas.
2. Take exercises applying these formulas.
3. Discuss polyhedral angles and the regular polyhedrons.

News Notes

It is announced officially that the next International Mathematical Congress will be held the first week in September, 1932, at Zurich, Switzerland. The Committee on Organization consists of Professor A. Fueter of the University of Zurich, president; Professors H. Fehr, Recteur of the University of Geneva, and M. Plancherel, of the Ecole polytechnique fédérale, Zurich, vice-presidents; and Professors F. Gonseth, of the Ecole polytechnique fédérale, and A. Speiser, of the University of Zurich, secretaries. The secretaries may be addressed at the Séminaire de mathématiques, Université, Zurich. It will be recalled that the first of these international mathematical congresses was held in Zurich in 1897.

Professor Henri Fehr, Head of the Department of Mathematics at the University of Geneva, has been appointed Recteur of the University. He is well known to Americans and to the mathematical and educational world at large as the editor of l'Enseignement Mathématique, and as Secretary of the International Commission on the Teaching of Mathematics, not only the present one, but the one that was founded in 1908. He was a student at the Sorbonne and at the Collège de France. He has held a professorship in the University of Geneva since 1900, and was twice Dean of the Faculty of Science. From 1928 to 1930 he was Vice-Recteur of the University, and has frequently represented it at various ceremonies at Oslo, Paris, Brussels, and elsewhere. He was one of the founders and was President of the Société Mathematique Suisse. He is also a member of the Légion d' Honneur, besides being corresponding member of various mathematical societies.

Professor W. D. Cairns of Oberlin was the speaker at the annual dinner of the Cleveland Mathematics Club on May 20, 1931. At this meeting the following officers were elected.

President, W. O. Smith, South High School.

Vice-president, Cora Lederer, Central High School.

Secretary, Alma Kerr, Thomas Jefferson High School.

Treasurer, W. M. Ruthledge, Lincoln High School.

Assuming the Congruence Theorems

By Joseph A. Nyberg

Hyde Park High School, Chicago

The suggestion is frequently made that the introduction to plane geometry can be simplified by assuming the first two congruence theorems, and that arguing in a circle can be eliminated by assuming all three of the congruence theorems. The teacher who believes in assuming all three does so because he believes it is better to assume

If the sides of one triangle are equal respectively to the sides of another triangle, the corresponding angles are equal

than to assume

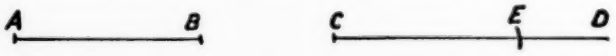
An angle has a bisector.

It is futile to discuss which is the better assumption although I believe that pupils, if given their choice, would prefer the second. This article will deal only with the question of assuming the first two theorems, in which case the third would afterwards be proved by using the base angles of an isosceles triangle. For convenience in comparing the methods I refer to a Class A, Teacher A, and Method A, when these theorems are assumed, and to a Class S, Teacher S, etc., when the theorems are proved by the traditional superposition method. My belief is that a visitor to the classes would be unable to tell whether he was in a Class A conducted by Teacher A or in a Class S conducted by Teacher S, that is, *the classroom procedure is the same in both classes*. If this statement can be proved, the arguments about the relative methods might be settled. I shall try to avoid the error of comparing the best type of one kind of teacher with a poorer one of the other type, and shall suppose each teacher uses the best technique possible.

Teacher A would prefer to avoid or to postpone superposition because he believes the pupils have difficulty in understanding the method. Let us examine what happens in the class and see if superposition *can* be avoided.

Before the topic of congruent triangles is reached, the class must discuss equal lines and equal angles. One of the first constructions consists in measuring off on CD a segment CE equal to another segment AB. And as a preparation for future exercises in which one

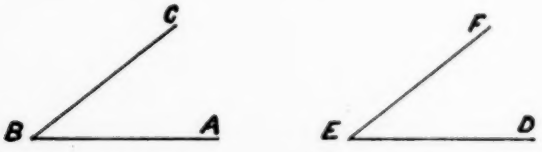
line is laid off another*, Teacher A discusses with the class the possible positions of the point E according as AB is greater than, less than, or equals to CD. This is equivalent to defining unequal lines. AB is less than CD if E lies between C and D; AB is greater than CD if



D lies between C and E. Clearly in this case there is no difference in the work of Teacher A and Teacher S; both must use superposition unless Teacher A decides to omit all theorems and exercises in which one line is measured off on another line

Consider next the matter of equal angles. The usual definition states that two angles are equal if they can be made to coincide. In order to make sure that the class really understands the definition Teacher A asks:

If ∠ABC is placed on ∠DEF so that B is on E and BA is along ED, where will BC fall if ∠B is less than ∠E? if ∠B is greater than ∠E? if ∠B equals ∠E?



Here we are defining unequal angles. This discussion is necessary because in later exercises and theorems we frequently copy an angle. In some cases a correct proof cannot be given unless we can state exactly the position of the terminal side of the angle. This discussion in class of unequal lines and angles is not intended as a preparation for the later topic of inequalities. It would take place even if the entire topic of inequalities were omitted from the course. The discussion belongs here solely for the purpose of giving the pupil a firm grasp of the definitions of equal lines and equal angles. Both Teacher A and Teacher S must include this discussion.

* The number of exercises in which one line is laid off another are so numerous that an illustration is unnecessary. In the theorem, "If two sides of a triangle are unequal, the opposite angles are unequal," the proof is based on measuring off the shorter line on the longer, or measuring off the longer on the prolongation of the shorter. Here the position of the end-point decides the rest of the proof.

Consider next the matter of congruent triangles. The first theorem which Teacher A prefers to assume is "Two triangles are congruent if two sides and the included angle of one are equal respectively to two sides and the included angle of the other." Even under Method A, which treats this statement as an axiom, the pupils cannot be expected to *apply* or to use the statement until they understand its meaning. Teacher A does not expect the class to use other axioms such as "If equals are added to equals, the sums are equal" without first discussing various illustrations, special cases of lines and angles, and other concepts that will make the axiom clear. Hence *congruence* must be discussed and illustrated. The usual definition states that figures are congruent if they can be made to coincide. But Teacher A knows that this is a mere shifting of words. *Coincide* must be explained, not only because it is useful in this instance but also because this word will be used in later exercises and theorems. Hence in order to make the assumed statement clear, Teacher A will very likely draw the figure on the board, and then ask a series of questions like:

Can we place AB on DE? What item in the hypothesis says that this is possible? If AB is on DE where will the line AC fall? Why? Where would AC fall if $\angle A$ were less than $\angle D$? If $\angle A$ were greater than $\angle D$? If AC is along DF why will C fall on F? Where would C fall if AC were less than DF? If AC were greater than DF? etc. Can we begin by placing AC on DF? by placing BC on EF?

But even these questions are not sufficient to explain *congruence* and *coincide*. Teacher A begins again with two other triangles KLM and RST, and asks the same questions all over again, or lets one pupil ask them and another pupil answer. With still other pairs of triangles various pupils tell how the triangles can be compared, until without any help from Teacher A, most of the pupils can compare the triangles.

Exactly the same discussion takes place in Class S under Teacher S since he, too, must explain *congruence* and *coincide*. After twenty minutes of such a discussion Teacher S would say that the theorem had been proved by superposition, and Teacher A would doubtless say "Ah, no! We have merely explained what congruence means so that the pupils can use the axiom intelligently." But a visitor in class would not be able to see the difference.

Suppose, however, that Teacher A avoids the words coincide and congruent and states the axiom by saying "Two triangles are *equal*,"

etc. thus using a word, *equal,* with which the pupil is familiar. Teacher A may say that the triangles are *duplicates* or *copies* which are alike in every respect. Since unfamiliar words are thereby avoided the class can without further discussion immediately apply the axiom to various exercises, selecting pairs of equal sides and equal included angles. This plan is very satisfactory as far as it goes, but let us examine the next stage of the work wherein the class is to use triangles to prove certain lines or angles are equal.

The class must use the statement "Corresponding parts of congruent triangles are equal." Much could be written about this simple statement but the essential items are that we (1) define corresponding parts, then (2) state that the corresponding parts are equal, and finally (3) advise the pupil as to the easiest way of finding which parts do correspond. On this basis we *define* corresponding parts as those which would coincide if the triangles are superposed. From this definition it follows that the corresponding parts are equal. And then, since we do not wish in each exercise to go through the trouble of superposing the triangles, we explain that the corresponding parts can be found by noticing which sides and angles are opposite each other. I do not see how Teacher A can present this explanation without using superposition. The only alternative would be to *define* corresponding sides as those which are opposite equal angles, and then to introduce an additional assumption: Corresponding parts are equal. Even with this alternative, some talk of superposition is inevitable because the reasonableness of the assumption must be explained. It must be justified as carefully as Teacher A justifies an axiom like "Things equal to the same thing are equal." Otherwise we shall have to believe that *any statement will be clear to a pupil if we merely call it an axiom.*

If the congruence theorems were used merely to prove triangles congruent and were not used to find the remaining corresponding parts, superposition might be avoided. However, since the corresponding parts are essential, Teachers A and S would need to present the same material in class. Teacher S would call it a proof; Teacher A would call it an explanation of coincidence. But I doubt if a visitor or a pupil could see the difference.

There are theorems (not like the congruence theorems) for which the classroom procedure is not the same when the theorem is assumed as when the theorem is proved. The best example is the theorem

about the sum of the angles of a triangle. If the statement is treated as an axiom, the pupil can get a thorough understanding of the statement by measuring the angles with a protractor. If the statement is to be proved the pupil does not use a protractor and the classroom work is decidedly different. With the congruence theorems, however, the classroom must be the same whether we assume or prove.

The favorite argument for assuming the theorems is that the pupil does not understand superposition proofs or finds these proofs difficult. The pupil's difficulty, according to these teachers, is shown by his unsatisfactory presentation of the proofs at the blackboard, the proofs sounding like memorization without understanding. Perhaps the trouble is not with the proof or with the pupil but merely that the teacher expects too much. Without the preliminary discussion about equal and unequal lines and angles, as presented in this article, the proof would seem peculiar to the pupil. The teacher must also expect to guide the pupil's presentation by frequent appropriate questions. The various possible positions of the lines must be discussed until the pupil sees the necessity for each item in the hypothesis. Writing the hypothesis on the board and then erasing each item as it is used is decidedly helpful. The proof can also be presented in narrative form as "What happened to A, B, and C." In most cases the difficulties are due to the fact that the introductory work has not been carefully done. "Why bother to discuss equal lines" thinks the inexperienced teacher, "Any dumbbell knows what we mean by equal lines." The pupil's troubles may merely mean that our technique for presenting the subject must be improved.

News Notes

At the Spring Meeting of the Connecticut Valley Section of the Association of Teachers of Mathematics in New England held in East Hampton, Massachusetts, on May 16, 1931, the following program was presented:

"Learning in First Year Algebra," by Nelson A. Jackson, Mount Hermon School.

"The New Type of Examination Set by the College Entrance Examination Boards": Algebra—Professor Bancroft H. Brown, Dartmouth College; "Geometry"—Robert R. Goff, Director of Mathematics, Senior and Junior High Schools, New Britain, Connecticut.

"Logic and Rigor in Trigonometry," by Professor Ralph D. Beetle, Dartmouth College.

"Some Sixteenth Century Number Theory," by Professor Suzan R. Benedict, Smith College.

NEW BOOKS

Beginner's Geometry. By Rolland R. Smith. The Macmillan Company, New York, 1928, 327 pages.

"There is no royal road to geometry, we all agree, but there should be some kind of a road at the beginning over which the pupil may walk with some assurance of interest and success and not a precipice which he must climb at the peril of his life."

The above quotation is from the preface to Smith's *Beginner's Geometry* and introduces one of the outstanding features of this carefully written and very teachable textbook. There is a beginning chapter of constructions, intuitively presented and interestingly written. This introduces the child to the concepts and ideas of geometry and illustrates effectively the need for and the function of the demonstration.

Another striking feature of the text is the use of "Developmental Exercises," as an approach to a new theorem, and, following each theorem, "Proficiency Exercises" which apply the theorem in varied situations. The "Developmental Exercises" develop the geometric ideas inductively and insure familiarity with an idea before a formal statement or demonstration of it is given. They provide an opportunity for the pupil to discover the general conclusion and consequently make the geometry more real. Most generalizations in life are discovered from experience with specific facts or cases.

The postulation of the first two congruence theorems also helps to make the book more teachable. Clearly the first objective of geometry is to teach children to prove exercises by the use of congruent triangles. In less progressive texts the proof of these first theorems by superposition provides a pattern for proofs which is at once abandoned in the exercises. The proof, therefore, is largely wasted. The superposition proof has also been a rather high "precipice" for most pupils to climb at the beginning of their course. For those pupils who need the proof by superposition for college entrance examinations, it is given in a later chapter where it belongs, if it belongs anywhere.

As a whole, the book impresses one as having been written by a master teacher and yet also by one so well trained in mathematics as to be unafraid of having some new ideas and of using old materials in new ways.

H. C. Christofferson
Miami University
Oxford, Ohio

Solid Geometry. By Rolland R. Smith and Leland W. Smith. The Macmillan Company, New York, 1930, 238 pages.

This book, like *Beginner's Geometry* by Rolland Smith, is built on the theory that geometry can be understood if properly presented. The presentation, according to the authors, is based on two principles: first, an analysis of the subject matter so that "'no element of instruction necessary to the understanding is omitted"; and second, the development of "new concepts before they are used in complex situations."

"Developmental Exercises" which precede each new proposition and concept are designed to prepare the way inductively for the generalization or idea

to be presented. Following each proposition or new topic are "Proficiency Exercises" which apply and extend the new ideas. Groups of "Miscellaneous Exercises" provide general application. The developmental exercises are an especially interesting feature of this book. Page 101-102, using pieces of chalk and a pile of cards to show that the volume of two prisms with equal bases and equal altitudes are equal, as an introduction to the postulation of Cavalieri's Theorem and the theorem for the volume of any pyramid, is an effective illustration of the "Developmental" technique used in a place where teachers of solid geometry will find it tremendously helpful.

The precise definitions of the areas and volumes of cylinders, cones, and sphere will be pleasing to mathematicians who have criticised high school texts for their very inadequate treatment of these topics. Furthermore, the liberal use of postulates to avoid proofs using that puzzling theorem of limits which seems to be so difficult for high school students to appreciate fully, should be satisfying to teachers.

A teacher who likes to see an abundance of applications to industry and life out of school will be somewhat disappointed. The exercises and applications are largely of the traditional type without a single photograph to show interesting uses for solid geometry facts and principles. This deficiency seems less significant, however, in view of the skillful organization of old materials and the very teachable presentation of the difficult spots found in solid geometry.

Teachers should welcome the precision and novelty, as well as the sensible attitude toward theorems involving limits, which is to be found in this new text.

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Diagnostic and Remedial Teaching in Arithmetic. By Dr. Leo J. Brueckner, Professor of Elementary Education, University of Minnesota, formerly Director of Instructional Research, Minneapolis Public Schools. The John C. Winston Co., Philadelphia, 1930, 348 pp. List, $2.00.

Dr. Brueckner has made a modern and timely contribution in his *Diagnostic and Remedial Teaching in Arithmetic,* a book that should be in every normal school and in the hands of every classroom teacher who is struggling with arithmetic results and is face to face with the necessity for removing failures.

It is recognized that only by proper diagnosis of pupils' errors, followed by systematic and carefully planned remedial teaching, can the difficulties in arithmetic teaching be overcome. This book tells the teacher how to make this diagnosis, both for her class and for individuals in it. It offers specific suggestions for remedial work and analyzes the most common faults of pupils. Its remedial procedures may be easily adopted in any classroom.

Particularly valuable is its analysis of the specific skills involved in whole numbers, fractions, decimals, and per cents. It lists the common pupil faults and presents definite remedial measures in each of these topics. The final chapters on problem solving contain a wealth of material relating to this important side of arithmetic teaching. The teacher can find there the latest thought on the selection of problems, methods of instruction, analysis of faults, and best remedial procedures.

Plane Trigonometry and Logarithms. By Thomas Marshall Simpson, Ph.D., University of Florida. Published by The John C. Winston Company, Philadelphia. List price, $1.52.

Plane Trigonometry and Logarithms has been developed from a course given for several years to freshmen at the University of Florida. During this time many practice tests have been given and analyzed in an attempt to learn from the students themselves which topics are difficult, which ones are most interesting, and which methods of presentation are simplest and most intelligible. Furthermore, the exact needs of the students in their future mathematical studies have been considered. As a result the text appears to be practical, teachable and interesting.

The approach is through the functional point of view. The trigonometric functions of the geometric angle are introduced in the beginning. The old unwieldy "reduction to the first quadrant" is eliminated and replaced by a simple and teachable device whereby the student reads results from a graph. The graphs of the function are constructed early in the course and used as a powerful tool, instead of being given later as an end in themselves.

Emphasis is upon "learning by doing." The pupil constructs brief tables of sines and cosines and immediately applies them to the solution of practical problems. He becomes interested at once in the usefulness of the subject.

Other helpful features are the numerous diagrams as an aid to visual instruction; carefully selected illustrative examples; a style and vocabulary suitable for the immature student; a fresh treatment of logarithms and interpolation; a clear and original presentation of the meaning of significant figures, and of "rounding off" numbers; absence of the disconcerting use of infinity as a number; flexible arrangement of material; sample modern tests on the longer chapters; and an amount of material such as can be covered in the time ordinarily devoted to trigonometry.

Instructional Tests and Chapter Tests for a First Course in Algebra. By Leonard D. Haertter, M.A., Head of the Department of Mathematics, John Burroughs School, St. Louis, Missouri; co-author, the *Engelhardt-Haertter Algebras.* Published by The John C. Winston Company, Philadelphia. List price, 56 cents. Teachers' Manual with Answers for above, 20 cents.

This new workbook provides a complete testing program for use with any modern, first-year algebra. The tests are diagnostic in character and are designed to reveal individual weaknesses, so that the necessary remedial work may be applied by the teacher.

There are both instructional tests on the separate skills, and chapter tests on the units, where these skills are combined. Thus, weaknesses are detected when the difficulties may be readily removed and before the wrong procedure has had a chance to become strongly fixed.

The tests are objective and easy to score—every answer is either entirely right or entirely wrong. Incidentally, provision is made for the student to score the results of his work.

Material of this sort is bound to save the teacher's time, and this particular workbook seems doubly valuable because it is not overloaded with non-essentials. Moreover, if the tests are given faithfully, and followed by the necessary remedial work, they will improve instruction almost as a matter of course. The Teachers' Manual contains detailed suggestions for use with various plans of instructions.